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Consequences of Charge Independence for the Magnetic Moments and Masses of Σ Hyperons*

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In this note we point out certain results on the magnetic moments and mass differences of the Σ^+, Σ^0, Σ^- hyperons consequent on the postulate of charge independence for strong interactions.

1. THEOREM ON ANOMALOUS MOMENTS FOR THE Σ TRIPLET

IN the currently accepted scheme of charge-independent interactions of the strongly coupled isotopic multiplets, the following assignments are made:

Nucleon	Doublet	$T=\frac{1}{2}$
π meson	Triplet	$T=1$
Σ hyperon	Triplet	$T=1$
K meson	Doublet	$T=\frac{1}{2}$
$\bar{K}$ meson	Doublet	$T=\frac{1}{2}$
Ξ hyperon	Doublet	$T=\frac{1}{2}$
Λ hyperon	Singlet	$T=0.$

The interaction Hamiltonian is to be a scalar ($T=0$) in isotopic spin space. The magnetic moment operator for an interacting system of particles is

$$M_3=\frac{ie}{2}\int dV\bigg[\bigg\{\bar{\psi}_N\frac{1+\tau_3}{2}(\mathbf{x}\times\gamma)_3\psi_N+\bar{\psi}_\Sigma\theta_3(\mathbf{x}\times\gamma)_3\psi_\Sigma+\bar{\psi}_\Xi\frac{\tau_3-1}{2}(\mathbf{x}\times\gamma)_3\psi_\Xi\bigg\}+\bigg\{(\mathbf{x}\times\nabla\phi_K^*)_3\frac{1+\tau_3}{2}\phi_K-\phi_K^*\frac{1+\tau_3}{2}(\mathbf{x}\times\nabla)_3\phi_K\bigg\}+\{(\mathbf{x}\times\nabla\phi_\pi^*)_3\theta_3\phi_\pi-\phi_\pi^*\theta_3(\mathbf{x}\times\nabla)_3\phi_\pi\}\bigg], \quad (1)$$

where γ is the Dirac operator, τ_3, θ_3 are the third components of the isotopic spin matrix for isotopic spins $T=\frac{1}{2}$ and $T=1$:

$$\tau_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \theta_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}.$$

ψ_N, ψ_Σ, ψ_Ξ stand for the baryon fields and ϕ_K, ϕ_π for the meson fields, and one has

$$\psi_N=\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix},\quad \psi_\Sigma=\begin{bmatrix}\psi_{\Sigma^+}\\ \psi_{\Sigma^0}\\ \psi_{\Sigma^-}\end{bmatrix},\quad \psi_\Xi=\begin{pmatrix}\psi_{\Xi^0}\\ \psi_{\Xi^-}\end{pmatrix},$$

$$\phi_\pi=\begin{bmatrix}\phi_{\pi^+}\\ \phi_{\pi^0}\\ \phi_{\pi^-}\end{bmatrix},\quad \phi_K=\begin{pmatrix}\phi_{K^+}\\ \phi_{K^0}\end{pmatrix}.$$

We have omitted the trivial term

$$\frac{ie}{2}\int\{\bar{\psi}_\Lambda\Theta(\mathbf{x}\times\gamma)_3\psi_\Lambda\}dV$$

for the Λ hyperon, where Θ is the isotopic spin matrix for $T=0$ which is the null 1×1 matrix.

The important point is to notice that the magnetic moment operator can be split into two components, one of which is a scalar and the other, the third component of a vector in isotopic spin space

$$M_3=S+V_3. \quad (2)$$

[Of course, the suffix 3 on the two sides of Eq. (2) refer to two different spaces.] This decomposition is a direct consequence of the expression for the electric charge in the form

$$Q=I_3+\tfrac{1}{2}U, \quad (3)$$

where U is the isofermion number.[1] In fact, the scalar and vector parts of M_3 originate from the same isotopic spin operators as does the charge Q.

Thus far we have not made use of any requirement of charge independence, but only of the assignment of the isotopic multiplets and of the expression for the charge. Let us now consider the expectation value of an operator of the type $(S+V_3)$ for an eigenstate of the total isotopic spin I and its third component I_3, i.e.,

$$\langle M_3\rangle=\langle I;I_3|S+V_3|I;I_3\rangle=\langle I;I_3|S|I;I_3\rangle+\langle I;I_3|V_3|I;I_3\rangle. \quad (4)$$

The first term of (4) simply gives the expectation value of the (isotopic scalar) operator S with respect to the space and spin part of the state. Making use of the

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[1] B. d'Espagnat and J. Prentki, Nuclear Phys. 1, 33 (1956).

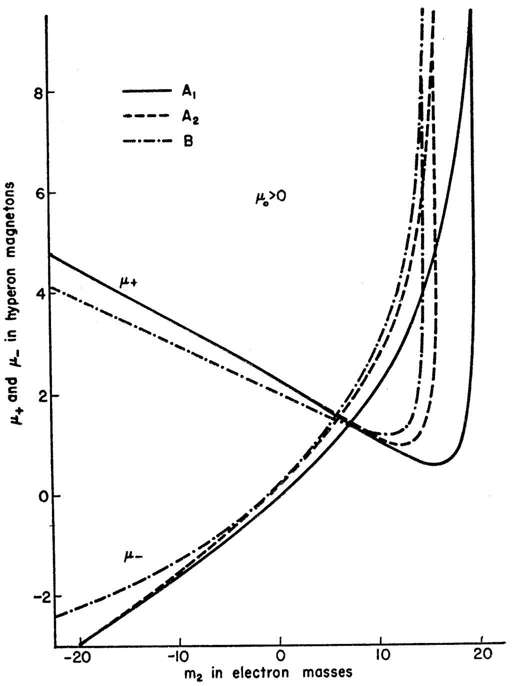

FIG. 1. μ_+ and μ_- as functions of m_2 (for observed value of m_1) for different cutoffs and $\mu_0>0$.

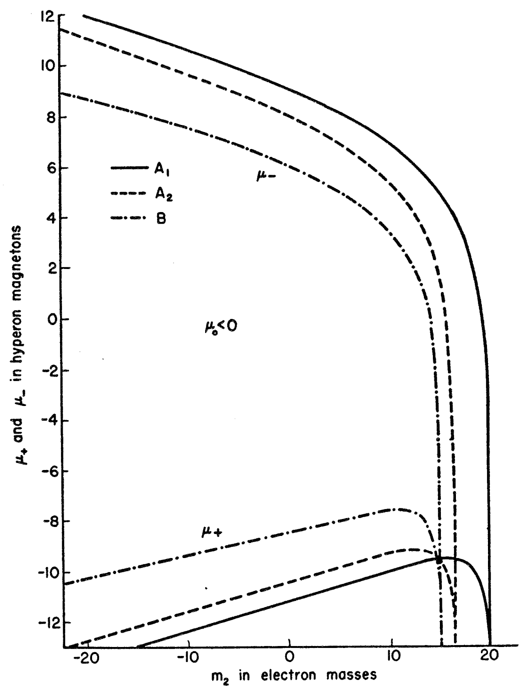

FIG. 2. μ_+ and μ_- as functions of m_2 for different cutoffs and $\mu_0<0$.

algebra of tensor operators, one can write

$$\langle I;I_3|S|I;I_3\rangle=\langle I\|S\|I\rangle,$$
$$\langle I;I_3|V_3|I;I_3\rangle=I_3\langle I\|V_3\|I\rangle, \tag{5}$$

where the reduced matrix elements $\langle I\|S\|I\rangle$, $\langle I\|V_3\|I\rangle$ are independent of I_3 and depend only on I and the space and spin parts of the states considered. Thus, making use of rotation invariance in isotopic spin space, we obtain for the expectation value of the magnetic moment operator for an isotopic multiplet

$$\mu(I,I_3)\equiv\langle I;I_3|M_3|I;I_3\rangle=S_I+V_II_3. \tag{6}$$

This conclusion is independent of any perturbation approximation.

In the framework of the d'Espagnat-Prentki formulation of the Gell-Mann-Nishijima scheme, the interactions preserve the strangeness quantum number in addition to the operators of baryon number N, electric charge, isofermion number U, the third component of isotopic spin I_3, and the total isotopic spin I. In fact, in view of the relation (3), if we choose our states so that they are eigenstates of I, I_3, N, and U, they will automatically be eigenstates of the electric charge Q also.

Let us now consider the triplet of states that correspond to the three physical Σ hyperons. Since all the above conditions are satisfied with

$$I=1,\quad U=0,\quad N=1,$$

we have

$$\mu_+=S+V,\quad \mu_0=S,\quad \mu_-=S-V, \tag{7}$$

where μ_+, μ_0, μ_- refer to the anomalous magnetic moments of the three hyperons and S and V are two functions of the masses and coupling constants of the various isotopic multiplets. From (7), we obtain

$$\mu_++\mu_-=2\mu_0. \tag{8}$$

This result is perturbation-independent and is exact if we neglect those interactions that violate charge independence. This relation is true not only for the frequency-independent part of the magnetic moment but for the complete magnetic moments including the form factors. It is interesting to note that (7) is actually true both for total and anomalous moments.

From the comments following Eqs. (5) and (6), it is clear that similar relations are applicable for more complicated systems like, say, the magnetic moments and electric quadrupole moments of nuclei belonging to a pure isotopic multiplet.

2. MASS SPECTRUM OF THE Σ TRIPLET

In an earlier communication,[2] we have discussed the mass difference of the charged Σ hyperons and have noted that the electromagnetic self-energies of these baryons are connected with their anomalous magnetic moments by equations of the type[3] (μ_+ is in hyperon

[2] E. C. G. Sudarshan and R. E. Marshak, Phys. Rev. **104**, 267 (1956).

[3] The third term has a negative sign, contrary to R. P. Feynman and G. Speisman, Phys. Rev. **94**, 500 (1954) and reference 2. The explanation is given by K. Huang, Phys. Rev. **101**, 1173 (1956). We are indebted to Dr. H. Katsumori, who called our attention to this error.

TABLE I. Values of a, b, and c for different cut-off functions.

Cutoff	$C(k^2)$	$G(k^2)$	a	b	c
$A1$	$2M^2/(2M^2-k^2)$	$2M^2/(2M^2-k^2)$	7.60	5.58	0.62
$A2$	$M^2/(M^2-k^2)$	$4M^2/(4M^2-k^2)$	5.96	5.13	0.62
B	$4M^2/(4M^2-k^2)^2$	$4M^2/(4M^2-k^2)$	6.45	5.41	0.84

magnetons)

$$\Delta(m_{\Sigma^+})=a-b\mu_+-c\mu_+^2,$$

where a, b, and c are certain quantities dependent on the cut-off functions and are defined by (α is the fine structure constant)

$$a=\left(\frac{\alpha m}{4\pi}\right)\frac{2}{\pi^2 i}\int\frac{d^4k(m+\mathbf{k})C(k^2)}{[(p-k)^2-m^2]mk^2},$$

$$b=\left(\frac{\alpha m}{4\pi}\right)\frac{3}{\pi^2 i}\int\frac{d^4kG(k^2)C(k^2)}{[(p-k)^2-m^2]m^2}, \tag{9}$$

$$c=\left(\frac{\alpha m}{4\pi}\right)\frac{3}{4\pi^2 i}\int\frac{d^4k(\mathbf{k}+\frac{3}{4}m)[G(k^2)]^2C(k^2)}{[(p-k)^2-m^2]m^3}.$$

The corresponding expressions for Σ^0 and Σ^- are[4]

$$\Delta(m_{\Sigma^0})=-c\mu_0^2,$$
$$\Delta(m_{\Sigma^-})=a+b\mu_--c\mu_-^2. \tag{10}$$

Values of a, b, and c for different cut-off functions which yield the correct neutron-proton mass difference[2] are listed in Table I (M is the nucleon mass).

We now propose to investigate the implications of the relation $2\mu_0=\mu_++\mu_-$ on the ordering of the observed masses of Σ^+, Σ^0, and Σ^-. For this purpose, let us introduce two quantities which can be evaluated from the *observed* masses, namely,

$$m_1=m_{\text{obs}}(\Sigma^-)-m_{\text{obs}}(\Sigma^+)=\Delta(m_{\Sigma^-})-\Delta(m_{\Sigma^+}),$$
$$m_2=\tfrac{1}{2}\{m_{\text{obs}}(\Sigma^+)+m_{\text{obs}}(\Sigma^-)\}-m_{\text{obs}}(\Sigma^0) \tag{11}$$
$$=\tfrac{1}{2}\{\Delta(m_{\Sigma^+})+\Delta(m_{\Sigma^-})\}-\Delta(m_{\Sigma^0}).$$

Using (9) and (10), we can rewrite

$$m_1=2\mu_0(b+2c\mu_1),$$
$$m_2=a-b\mu_1-c\mu_1^2, \tag{12}$$

where

$$\mu_1=\tfrac{1}{2}(\mu_+-\mu_-)=\mu_+-\mu_0=\mu_0-\mu_-. \tag{13}$$

If we use a weighted average of the observed masses[5] of Σ^+ and Σ^-, we find that m_1 is rather accurately known, namely,

$$m_1=(15.7\pm1.9)m_e.$$

On the other hand, because of the inaccuracy in the mass of Σ^0, we have

$$m_2=(8\pm7)m_e.$$

We have therefore fixed m_1 (at 15.7) in Eqs. (12) and solved for μ_0 and μ_1 (and hence μ_+ and μ_-) as functions of m_2. Set I of the solutions for μ_+ and μ_- (corresponding to $\mu_0>0$) is plotted in Fig. 1 for the three different cutoffs of Table I; Fig. 2 contains Set II corresponding to $\mu_0<0$.

From Figs. 1 and 2, it is seen that μ_+ possesses an extreme value (which is a minimum when $\mu_0>0$ and a maximum when $\mu_0<0$), when m_2 approaches the positive limit of its experimental error. The quantity μ_- is a monotonic function of m_2; the situation would actually be reversed for $m_1<0$. Table II lists the values of μ_0, μ_+, and μ_- predicted for $m_2=8$ and Table III the extreme values of μ_+ with their associated values μ_-, μ_0, and m_2—for each of the three cutoffs. We consider the set of solutions corresponding to $\mu_0>0$ as more probable because the magnitudes of μ_+ and μ_- are much smaller. Indeed, the signs as well as the magnitudes of μ_+ and μ_- given in Fig. 1 are much more plausible on the basis of a meson-theoretic origin.[2] Direct measurements of the anomalous moments of the Σ hyperons, and especially the sign of the Σ^0 anomalous moment, would be most interesting.

TABLE II. Anomalous moment parameters from observed masses.

Cutoff	m_2	Set I μ_0	Set I μ_+	Set I μ_-	Set II μ_0	Set II μ_+	Set II μ_-
$A1$	8	1.44	1.33	1.55	−1.44	−10.33	7.45
$A2$	8	1.71	1.30	2.12	−1.71	− 9.50	6.08
B	8	2.58	1.17	3.99	−2.58	− 7.65	2.49

TABLE III. Values of parameters for turning point in μ_+.

Cutoff	m_2	Set I μ_+	Set I μ_-	Set I μ_0	Set II μ_+	Set II μ_-	Set II μ_0
$A1$	15.9	0.5	4.5	2.5	−9.5	4.5	−2.5
$A2$	12.5	0.9	4.1	2.5	−9.1	4.1	−2.5
B	11.2	1.1	3.2	2.2	−7.6	3.2	−2.2

3. GENERAL REMARKS

Relation (8) is independent of the space-time transformation properties (including parity) of the isotopic multiplets. In writing down the explicit expression (1), we have assumed that the baryons are particles of spin $\frac{1}{2}$ and the mesons are particles of spin 0. But this restriction can be relaxed and (2) would still be correct. In other words, as long as the interactions are charge-independent, all the results of Sec. 1 are correct. On the contrary, the results of Sec. 2 hold only for baryons of spin $\frac{1}{2}$.

[4] *Note added in proof.*—In the expression (10) for $\Delta(m_{\Sigma 0})$, we haev neglected a contribution $-c'\mu_0'^2$ from the diagram corresponding to the virtual processes $\Sigma_0\to\Lambda_0+\gamma\to\Sigma_0$. If Λ and Σ have the same parity and coupling constants, $\mu_0'\simeq\mu_0$; also $c'\simeq c$ if we neglect the Σ, Λ mass difference. Hence the rhs of (10) for $\Delta(m_{\Sigma 0})$ should be multiplied by a factor ~2. However, as can be seen from Table I, c is quite small compared with a and b and m_2 would be changed only by ~1.5 m_e.

[5] Plano, Samios, Schwartz, and Steinberger, U. S. Atomic Energy Commission Report NYO-4715 (unpublished); Fry, Schneps, Snow, Swami, and Wold, Phys. Rev. **104**, 270 (1956); Alvarez, Bradner, Falk, Gow, Rosenfeld, Solnitz, and Tripp, preliminary University of California Radiation Laboratory Report (unpublished).