ADDITIVE AND MULTIPLICATIVE QUANTUM NUMBERS

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1. Introduction. In the current formalism of elementary particle physics, a single particle state is characterised by a set of quantum numbers some of them taking on only a discrete set of values (like charge, spin, etc.) while some others take on any of a set of continuous eigenvalues (like momentum, energy, etc.). In the passage from one particle states to many particle states, the various quantum numbers of the many particle states may be formed out of those of the one particle states generally by simple addition. For example the total charge or momentum is the sum of the corresponding quantities for one particle states.

However, there exists a distinct class of quantum numbers whose law of composition is multiplicative ; and these are the quantum numbers associated with the discontinuous operations of the various symmetry groups admitted by the fields under consideration. The more imporant examples are the quantum numbers of parity (space reflection) and charge parity (charge conjugation). It is the purpose of this note to point out the general correspondence between additive and multiplicative quantum numbers and to investigate the physical implications of stating an empirical conservation law (like " strangeness " conservation) in a multiplicative or additive form.

2. Additive and Multiplicative Operators. We shall start with the elementary remark that to every operator A with eigenvalues a_i for which the law of composition (for many particle states) is additive, there corresponds the operator U :

$$U = \exp(\lambda A)$$

with eigenvalues u_i

$$u_i = \exp(\lambda a_i)$$

for which the law of composition is multiplicative. Since U is a function of A, U and A can be made simultaneously diagonal. The parameter λ is completely arbitrary. If we now choose λ to be pure imaginary and a submultiple of $(2\pi i)$, U will be idempotent (i.e. $U^n = I$ for some finite n) and hence U can be interpreted to be a "parity".

If we now consider, simultaneously, more than one quantum number, the preceding correspondence can be extended. Thus to a set of operators $\{A^{(i)}\}$ with eigenvalues $\{a_j^{(i)}\}$ combining additively there corresponds a set of operators $\{U^{(i)}\} = \{\exp[\lambda_i A^{(i)}]\}$ with eigenvalues $\{u_j^{(i)}\} = \{\exp[\lambda_i a_j^{(i)}]\}$ combining multiplicatively. Here again the set of parameters $\{\lambda_i\}$ is completely arbitrary. We may now choose any one of these to be a submultiple of $(2\pi i)$ and interpret the corresponding multiplicative quantum number to be a "parity".

3. The Gödel Operator. This purely algebraic correspondence can be made use of in writing a single operator G and quantum number g in place of the (ordered) set of quantum numbers using Gödel numbers. For this purpose, label the operators (and corresponding eigenvalues) in some arbitrary, but definite, order; now choose λ_i to be the logarithm of the ith prime number, so that

$$e^{\lambda_1} = 3,\ e^{\lambda_2} = 5,\ e^{\lambda_3} = 7,\ e^{\lambda_4} = 11, \text{ etc.}$$

Then the set of quantum numbers $\{a^{(1)}, a^{(2)}, a^{(3)}, \ldots\}$ is equivalent to the single quantum number

$$g = 3^{a(1)}\, 5^{a(2)}\, 7^{a(3)} \ldots$$

It is easily verified that this correspondence is unique; and that conservation of each of the quantum numbers $a^{(1)}, a^{(2)}, \ldots$ is equivalent to the conservation of the single Gödel number g. An equivalent Gödel number g' can be got by replacing the exponents in the defining equation for g by any linearly independent combinations of the $a^{(i)}$. This corresponds to the well-known result that if α, β refer to two operators whose eigenvalues are conserved, $\lambda\alpha + \mu\beta$ also corresponds to an operator with conserved eigenvalues for all values of the numbers λ, μ. Conservation of g' is completely equivalent to the conservation of g.

Naturally the Gödel number for a many particle states is the product of Gödel numbers of the one particle states. Conservation of g takes care of all conservation laws corresponding to additive quantum numbers. A single multiplicative quantum number π can be included in this scheme by multiplying the Gödel operator G by the multiplicative operator π to give G' :

$$G' = G\pi \rightleftarrows g' = g\pi .$$

Then conservation of G' is equivalent to all conservation laws considered.

4. The Gödel Matrix. This transcription fails if one has more than a single "parity" that is conserved in any interaction. In such a case the scheme can be extended by replacing the Gödel number g by a Gödel matrix. The matrix $\mathfrak{g}$ would correspond to an operator matrix $\mathfrak{g}$. The matrix $\mathfrak{g}$ is got by multiplying by g a matrix which is the direct (Kronecker) product of a group of idempotent matrices. Each matrix factor would correspond to the eigenvalue of a unimodular multiplicative ("parity ") quantum number. This is simply an extension of the Gödel scheme in which we include in addition to prime number factors also idempotent matrices. Thus

$$\mathfrak{g} = g\, P_1 \# P_2 \# P_3 \# \ldots ,$$

where $P_1, P_2, P_3, \ldots$ are related to the conserved multiplicative quantum numbers π_i by the mapping of powers of idempotent matrices P (with $P^n = I$) on the nth roots of unity.

One can now pass from " strong " interactions (which conserve all the quantum numbers) to weak interactions (which conserve only some of them) by choosing a reduced Gödel number $\mathfrak{g}'$ got from $\mathscr{G}$ by substituting unity (or the unit matrix I) for those factors of $\mathfrak{g}$ corresponding to non conserved quantities, and requiring the conservation of only this reduced Gödel number $\mathfrak{g}'$.

5. Physical Interpretation. We may now consider the physical interpretation of the mathematical formalism. As mentioned earlier physically interpretable multiplicative quantum numbers are those corresponding to the disjoint operations of a continuous group; and conservation of such a quantum number is equivalent to the conservation of the symmetry under this disjoint operation. One would hence associate a multiplicative operator (i.e. one whose eigenvalue for a many particle states is got by multiplication of the corresponding single particle eigenvalues) with symmetry under a disjoint operation. This in fact is the case. To demonstrate this, it is sufficient to recall that if A is the generator of infinitesimal displacements, the operator

$$U = e^{\lambda A}$$

is the generator of a finite displacement λ. For example, P_x generates infinitesimal translations along the x direction; then $e^{\lambda p_x}$ generates a finite displacement λ along the x direction. Hence conservation of a Gödel number g is equivalent to a lattice type symmetry of the system considered under a series of finite displacements. The prime factors correspond to the " lattice constants ."

A similar interpretation can be given for the matrix factors of the Gödel matrix $\mathfrak{g}$. These correspond to finite 2 dimensional rotations and their conservation is equivalent to a "wheel" type symmetry; the index of idempotency n is the number of " spokes " of the wheel. The connection between multiplicative unimodular operators and rotational symmetries is thus evident.

Since a finite displacement can be built up of infinitesimal displacements but not vice versa, the Gödel scheme can be made less restrictive than the usual scheme which demands invariance under infinitesimal displacements and/or rotations (and thus the transition to a particular Gödel scheme would permit a "structure" to the base space.) The usual requirement of invariance under infinitesimal displacements would be equivalent to the conservation requirement for all possible assignment of the Gödel numbers.

We could, as well, have assigned abstract elements of a multiplicative group by replacing each factor by the generating element of a (distinct) group and consider the group element formed by the direct product of these elements. The idempotent matrix factors would then correspond to cyclic groups. Since all the operators are simultaneously diagonal they commute and hence the corresponding abstract group should be Abelian (commutative). We shall however omit this transcription in view of the physical interpretation given above.

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