

A Class of Solvable Potentials(*)

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Summary. — A systematic method of constructing (velocity-independent) potentials, for which the s -wave Schrödinger equation can be solved in terms of known functions, is presented. Several such examples are constructed and the analytic structure of some of the corresponding scattering amplitudes worked out in detail. The use of Darboux's theorem allows a significant extension of the class of solvable potentials.

1. — Introduction.

There has been a revival of interest in the structure of scattering amplitudes and wave functions in quantum mechanics, largely due to the method of dispersion relations as a computational aid in the theory of strong interactions. In most applications of dispersion relations, analytic properties are used to furnish numerical approximation; and it is always desirable to ascertain the nature of the exact solutions at least in simpler models. Solvable potentials furnish precisely this kind of model. It is perhaps surprising that the number of solvable potentials is so few: for the most part the explicit examples were isolated discoveries. A notable exception is the work of BARGMANN⁽¹⁾ who has shown how to construct families of solvable potentials leading to scattering amplitudes which are rational functions of the square root of the energy.

Bargmann's method cannot be extended to more complicated functional dependence of the scattering amplitude in any obvious manner; the so-called potentials of the linear type and potentials of the quadratic type have res-

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(1) V. BARGMANN: *Rev. Mod. Phys.*, **21**, 488 (1949).

pectively one and two poles in general. There is however another known potential ⁽²⁾, namely the exponential potential, where the scattering amplitude has an infinite number of so-called «redundant» poles. In the course of a study of solvable potentials undertaken at the suggestion of the second author, JORDAN ⁽³⁾ discovered that the redundant poles correspond to the circumstance that for integral values of the Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ are not linearly independent; and it was suggested that this circumstance could be generalized to a wider class of second order differential equations of the hypergeometric family.

The present investigation implements this suggestion. We look for simple functional transformations in the hypergeometric, the confluent hypergeometric and the Bessel differential equations which reduce them to the form of a radial Schrödinger equation for the s -wave. For every such transformation we then know the complete analytic solution; and by studying the asymptotic form the s -matrix may be evaluated. If $f(\pm k, r)$ are the solutions of the radial Schrödinger equation (with units chosen so that $E = k^2$) for the s -wave:

$$\frac{d^2}{dr^2} \varphi(r) + k^2 \varphi(r) = V(r) \varphi(r),$$

with the property

$$\lim_{r \rightarrow \infty} \exp[\pm ikr] f(\pm k, r) = \lim_{r \rightarrow \infty} \chi(\pm k, r) = 1$$

then the continuum-normalized solution obeying the proper boundary condition (i.e. $\varphi(0) = 0$) is given by ⁽⁴⁾

$$\varphi(k; r) = \frac{f(k, 0)f(-k, r) - f(-k, 0)f(k, r)}{2i|f(k, 0)|}$$

Since the asymptotic form of $\varphi(k; r)$ is

$$\varphi(k; r) \sim \frac{f(k, 0) \exp[ikr] - f(-k, 0) \exp[-ikr]}{|f(k, 0)|}$$

we obtain the expression for the s -matrix in the form:

$$s(k) = \exp[2i\eta(k)] = f(k, 0)/f(-k, 0).$$

⁽²⁾ S. T. MA: *Phys. Rev.*, 69, 668 (1946); 71, 195 (1947); D. TER HAAR: *Physica* 12, 509 (1946).

⁽³⁾ T. F. JORDAN: private communication.

⁽⁴⁾ Compare: R. JOST: *Helv. Phys. Acta*, 20, 256 (1947).

Unlike BARGMANN who was interested in exhibiting families of phase equivalent potentials, we shall not be interested in the Jost functions $\varphi(\pm k; r)$ but only in the s -matrix; we shall consequently ignore absolute normalizations.

While we present explicit forms for a class of solvable potentials, we would like to consider that the chief point of interest should be the methodology of the present investigation. Clearly the number of explicit examples, even the ones derived from the hypergeometric family, could be multiplied, by considering a wider variety of transformations and by the use of Darboux's theorem⁽⁵⁾. In Section 2 we discuss the transformation on the hypergeometric equation and consider a special class of solvable potentials so obtained; a particular choice of the parameters in this case leads to one of Bargmann's potentials of the linear type. Section 3 discusses another class of transformations on the hypergeometric equation; and we illustrate the computation of the s -matrix in this case, which has an infinite number of poles. Sections 4 and 5 deal with transformations of the confluent hypergeometric equation and the Bessel equation respectively. Some remarks of a general nature are made in the concluding section.

2. - Transformations on the hypergeometric equation.

Consider the general linear second order differential equation in one variable:

$$\frac{d^2}{dz^2} u(z) + p(z) \frac{d}{dz} u(z) + q(z)u(z) = 0.$$

On making the substitutions

$$z = f(r), \quad u(z) = g(r)\varphi(r), \quad g(r) \neq 0$$

we obtain the transformed equation

$$\frac{d^2}{dr^2} \varphi(r) + A(r) \frac{d}{dr} \varphi(r) + B(r) \varphi(r) = 0,$$

where

$$(2.4) \quad \begin{cases} A(r) = \frac{2}{g(r)} \frac{d}{dr} g(r) + P(r) \frac{d}{dr} f(r) - \frac{(d^2 f(r)/dr^2)}{(d/dr) f(r)}, \\ B(r) = \frac{d^2}{dr^2} g(r)/g(r) + Q(r) \left\{ \frac{d}{dr} f(r) \right\}^2 + \left\{ \frac{dg(r)/dr}{g(r)} \right\} \left\{ P(r) \frac{df(r)}{dr} - \frac{d^2 f(r)/dr^2}{df(r)/dr} \right\} \\ P(r) = p \{f(r)\} \\ Q(r) = q \{f(r)\} \end{cases}$$

⁽⁵⁾ See, for example E. L. INCE: *Ordinary Differential Equations* (New York, 1956)

For (2.3) to be of the form of the radial Schrödinger equation (1.1) it is necessary and sufficient that

$$A(r) = 0; \quad B(r) = k^2 + V(r); \quad \frac{\partial}{\partial k} V(r) = 0$$

For the particular case of the hypergeometric equation (*)

$$P(r) = \frac{c - (a + b + 1)f(r)}{f(r)\{1 - f(r)\}}$$

$$Q(r) = -\frac{ab}{f(r)\{1 - f(r)\}};$$

with this form of the functions $P(r)$, $Q(r)$ the differential equation $(Ar) = 0$, eq. (2.5), can be integrated once to give

$$\frac{d}{dr} f(r) = Mg^2(r) f^c(r) \{1 - f(r)\}^{a+b-c+1}$$

where M is an integration constant which is so far arbitrary. Using this result, the requirement $B(r) = k^2 + V(r)$ can be cast in the form of a third order non-linear differential equation

$$\frac{f'''}{f'} - \frac{1}{2} \left(\frac{f''}{f'}\right)^2 + \left\{ \frac{c - \frac{1}{2}c^2}{f^2} + a + b + 1 - \frac{c}{(1-f)^2} - \frac{2ab - (a+b+1-c)c}{f(1-f)} \right\} f'^2 + 2V + 2k^2 = 0$$

where the primes denote differentiation. Since this general solution of this differential equation is intractable, we must find particular solutions *i.e.* particular functions $f(r)$, $g(r)$ which lead to the Schrödinger form (1.1).

We are thus led to consider particular choices for $f(r)$ and $g(r)$ so as to obtain solvable potentials. Consider the choice

$$\frac{f'^2}{f(1-f)} = 4\alpha^2 = \frac{k^2}{ab},$$

where ab is an arbitrary real parameter; this equation can be integrated to give

$$f(r) = \sin^2(\alpha r + \beta)$$

(*) Here, and in the sequel, we use the notation and results from A. ERDELYI *Higher Transcendental Functions*, vol 1 (New York, 1953).

where β is an integration constant. Using (2.6) we obtain

$$g^2 = \frac{2\alpha}{M} \sin^{1-2c}(x\alpha + \beta) \cos^{1-2(a+b-c)}(x\alpha + \beta),$$

so that

$$\frac{g'}{g} = \alpha \left\{ (a+b-c+\frac{1}{2}) \operatorname{tg}(x\alpha + \beta) - (c-\frac{1}{2}) \operatorname{ctg}(x\alpha + \beta) \right\}$$

With this choice of $f(r)$ the defining relation (2.5) for $V(r)$ can be rewritten in the form

$$V(r) = \frac{g'}{g} \left[\left(\frac{g'}{g} \right)^2 = \alpha^2 \left[(a+b)^2 + \frac{\frac{1}{4} - (a+b-c)^2}{\cos^2(x\alpha + \beta)} - \frac{(c-\frac{1}{2})(c-\frac{3}{2})}{\sin^2(x\alpha + \beta)} \right] \right]$$

To obtain a *velocity-independent* potential we choose α and $a+b$ to be independent of k so that

$$a = \gamma + \sqrt{\gamma^2 + k^2/4x^2} = \gamma + \frac{\kappa}{2\alpha},$$

$$b = \gamma - \sqrt{\gamma^2 + k^2/4x^2} = \gamma - \frac{\kappa}{2\alpha},$$

and the transformed equation is of the form

$$\frac{d^2\varphi(r)}{dr^2} + \left[\kappa^2 + \alpha^2 \left[\frac{\frac{1}{4} - (2\gamma - c)^2}{\cos^2(x\alpha + \beta)} - \frac{(c-\frac{1}{2})(c-\frac{3}{2})}{\sin^2(x\alpha + \beta)} \right] \right] \varphi(r) = 0,$$

where we have introduced the new potential

$$V(r) = \frac{\alpha^2}{4} \left[\frac{1 - 4(2\gamma - c)^2}{\cos^2(x\alpha + \beta)} - \frac{(2c-1)(2c-3)}{\sin^2(x\alpha + \beta)} \right]$$

(normalized to zero at infinity) and the asymptotic momentum $\kappa = \sqrt{k^2 + 4x^2\gamma^2}$. The general solution to this equation is of the form:

$$\varphi(r) = \operatorname{tg}^{c-\frac{1}{2}}(x\alpha + \beta) \cos^{2\gamma}(x\alpha + \beta) \cdot \{ A F_1(a, b; c; \sin^2(x\alpha + \beta)) + B F_2(a, b; c; \sin^2(x\alpha + \beta)) \},$$

with A, B constants to be so determined that $\varphi(0) = 0; \varphi'(0) = 1$

For the choice

$$\gamma = 0, \quad \beta = \frac{\pi}{2}, \quad c = \frac{1}{2}, \quad a = -b = \frac{i\kappa}{\alpha},$$

and by $-i\alpha$ in place of α everywhere we get the potential

$$V(r) = \frac{2\alpha^2}{\cosh^2 \alpha r}$$

which is recognized as one of Bargmann's potentials of the linear type. We also have, using the explicit forms for the hypergeometric functions, the remarkably simple expression

$$(2.10) \quad \varphi(r) = \left\{ \frac{1}{\alpha} \sin \kappa r + \frac{1}{\kappa} \operatorname{tgh} \alpha r \cos \kappa r \right\},$$

with the asymptotic form

$$\varphi(r) \sim \frac{1}{2\alpha r} \{ (\alpha - i\kappa) \exp[-i\kappa r] + (\alpha + i\kappa) \exp[i\kappa r] \}$$

The s -matrix is then given by

$$s(\kappa) = \frac{\kappa - i\alpha}{\kappa + i\alpha},$$

in agreement with the result of Bargmann. This repulsive potential hence corresponds to an s -matrix with a single «redundant pole» (*) for $\kappa = -i\alpha$ in the lower half plane and a zero which is its mirror image.

3. - Another solvable potential.

Let us now consider another particular transformation on the hypergeometric equation:

$$g(r) = \exp[i\kappa r]$$

so that (2.6) gives the differential equation

$$\frac{d}{dr} f(r) = M \exp[2i\kappa r] \left(\frac{f}{1-f} \right)^c (1-f)^{a+b+1}$$

The potential function $V(r)$ becomes

$$V(r) = abM^2 \exp[4i\kappa r] \left(\frac{f}{1-f} \right)^{2c-1} (1-f)^{2(a+b+1)} \frac{abf^a}{f(1-f)}.$$

To make $V(r)$ independent of k the simplest choice is to make $f(r)$ and ab

independent of k ; in this case (3.2) leads to the result

$$f(r) = \exp[\alpha r]; \quad c = \frac{\beta}{\alpha} \frac{2ik}{\alpha}$$

where α, β are independent of k . Substituting in (3.2) we must have following as identities in r :

$$M \exp[\beta r] (1 - \exp[\alpha r])^{a+b+1} + \frac{2ik - \beta}{\alpha} \alpha \exp[\alpha r]$$

Hence

$$\alpha = \beta \quad M; \quad a + b \quad \frac{2ik}{\alpha} = c - 1,$$

so that the potential becomes:

$$V(r) = \frac{ab\alpha^2 \exp[\alpha r]}{1 - \exp[\alpha r]}$$

Writing $ab = \gamma$ (independent of k) and $\lambda = \gamma\alpha^2$ we obtain the result that the radial Schrödinger equation

$$(3.3) \quad \frac{d^2}{dr^2} \varphi(r) + \left(k^2 - \frac{\lambda \exp[\alpha r]}{1 - \exp[\alpha r]} \right) \varphi(r) = 0$$

has the general solution

$$(3.4) \quad \varphi(r) = \exp[ikr] F(a, b; c; \exp[\alpha r]),$$

where $F(a, b; c; z)$ is the general solution of the hypergeometric equation with the parameters

$$(3.5) \quad \begin{cases} a = \frac{i}{\alpha} (\sqrt{k^2 + \lambda} - k), \\ b = -\frac{i}{\alpha} (\sqrt{k^2 + \lambda} + k), \\ c = 1 - \frac{2ik}{\alpha} \end{cases}$$

Since $a + b - c + 1 = 0$, this is a degenerate case of the hypergeometric equation (?).

The two linearly independent solutions $F(a, b; c; \exp[\alpha r])$ in the degener-

ate case are respectively

$$\begin{aligned}
 u_1(r) &\equiv F_1(a, b; a + b + 1; \exp[\alpha r]) = \frac{\Gamma(1 + a + b)}{\Gamma(1 + a)\Gamma(1 + b)} - \\
 &- (1 - \exp[\alpha r]) \frac{\Gamma(1 + a + b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(1 + n + a)\Gamma(1 + n + b)}{\Gamma(1 + a)\Gamma(1 + b)\Gamma(n + 1)\Gamma(n + 2)} \cdot \\
 &\quad \cdot \{k_n - \log_e(1 - \exp[\alpha r])\} (1 - \exp[\alpha r])^n, \\
 &= F_2(a, b; a + b + 1; \exp[\alpha r]) = \exp[-(a + b)\alpha r] \left[\frac{\Gamma(1 - a - b)}{\Gamma(1 - a)\Gamma(1 - b)} - \right. \\
 &- (1 - \exp[\alpha r]) \frac{\Gamma(1 - a - b)}{\Gamma(-a)\Gamma(-b)} \sum_{n=0}^{\infty} \frac{\Gamma(1 + n - a)\Gamma(1 + n - b)}{\Gamma(1 - a)\Gamma(1 - b)\Gamma(n + 1)\Gamma(n + 2)} \cdot \\
 &\quad \left. \cdot \{k'_n - \log_e(1 - \exp[\alpha r])\} (1 - \exp[\alpha r])^n \right],
 \end{aligned}$$

where

$$\begin{aligned}
 k_x &= \frac{d}{dx} \log_e \left\{ \frac{\Gamma(x + 1)\Gamma(x + 2)}{\Gamma(x + 1 + a)\Gamma(x + 1 + b)} \right\}, \\
 k'_x &= \frac{d}{dx} \log_e \left\{ \frac{\Gamma(x + 1)\Gamma(x + 2)}{\Gamma(x + 1 - a)\Gamma(x + 1 - b)} \right\}.
 \end{aligned}$$

The admissible continuum-normalized solutions of (3.4) are:

$$\varphi(r) = \exp[ikr] \frac{u_2(0)u_1(r) - u_1(0)u_2(r)}{u_2(0)u'_1(0) - u'_2(0)u_1(0)}$$

$$u_1(0) = \frac{\Gamma(1 + a + b)}{\Gamma(1 + a)\Gamma(1 + b)}; \quad u_2(0) = \frac{\Gamma(1 - a - b)}{\Gamma(1 - a)\Gamma(1 - b)},$$

we have, apart from a constant, the explicit form for $\varphi(r)$.

The asymptotic form of the wavefunction can easily be calculated; since $V(r) \rightarrow 0$ for $r \rightarrow \infty$, this asymptotic form may be written

$$\text{const } (s(k) \exp[-ikr] + \exp[ikr])$$

with the s -matrix in the explicit form

$$s(k) = \frac{1 - \frac{\lambda}{\alpha^2} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - a)\Gamma(n + 1 - b)k'_n}{\Gamma(1 - a)\Gamma(1 - b)\Gamma(n + 1)\Gamma(n + 2)}}{1 - \frac{\lambda}{\alpha^2} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 + a)\Gamma(n + 1 + b)k_n}{\Gamma(1 + a)\Gamma(1 + b)\Gamma(n + 1)\Gamma(n + 2)}}$$

Using the identity (7)

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1+a)\Gamma(n+1+b)}{\Gamma(1+a)\Gamma(1+b)\Gamma(n+1)\Gamma(n+2)} \{k_n - \log_e(1-z)\} (1-z)^n$$

$$= \left\{ \frac{\Gamma(1+a+b)}{\Gamma(1+a)\Gamma(1+b)} - F_1(a, b; 1+a+b; z) \right\} \frac{\Gamma(a)\Gamma(b)}{\Gamma(1+a+b)} (1-z)^{-1}$$

and a related formula involving k'_n we can rewrite the expression for $s(k)$ in the form

$$s(k) = \frac{1}{1} \frac{\lambda \int \frac{1}{ab} \frac{\Gamma(-a)\Gamma(-b)}{\Gamma(1-a-b)} F_1(a, -b; 1$$

$$1 \frac{\lambda \int \frac{1}{ab} -$$

Using the result $f_1(a, b; c; 0) = 1$ and recalling $ab = \lambda/\alpha^2$ we get the final form for the scattering matrix:

$$s(k) = \frac{\Gamma(1+a+b)}{\Gamma(1-a-b)} \cdot \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(1+a)\Gamma(1+b)} \frac{\beta(-a, -b)}{\beta(a, b)}$$

where

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

The singularities of this s -matrix can be easily determined since the analytic properties of the γ -function are completely known; $\Gamma(z)$ is meromorphic in the entire z -plane with simple at $z=0, 1, \dots$ and vanishes nowhere. The factors

$$\Gamma(1-a-b) \equiv \Gamma\left(1 - \frac{2ik}{\alpha}\right); \quad \Gamma(1+a+b) \equiv \Gamma\left(1 + \frac{2ik}{\alpha}\right),$$

give rise to an infinite sequence of purely imaginary zeroes $k = in\alpha/2$ and poles $k = -in\alpha/2$, where n is any positive integer. Note that all these poles are «redundant» and that the poles and zeroes are mirror images of each other in the real axis in the k -plane.

The quantities a, b have square root branches considered as functions of k ; but since the factors $\Gamma(1-a), \Gamma(1-b)$ occur symmetrically they together contain no branch cuts. The function $\Gamma(1-a) \cdot \Gamma(1-b)$ considered as a function of k has a sequence of simple poles for

$$k \mp \sqrt{k^2 + \lambda} = in\alpha; \quad n = 1, 2, \dots$$

i.e. for the purely imaginary values

$$k = i \frac{\lambda + n^2 \alpha^2}{2nr} \quad i \frac{\lambda + n^2 \alpha^2}{2n|\alpha|}$$

which tend to coincide with the poles of the factor $\Gamma(1 + a + b)$ as $n \rightarrow \infty$. The factor $\Gamma(1 + a) \cdot \Gamma(1 + b)$ gives sequence of simple zeroes for the s -matrix, (which are the mirror images of these poles). Note that all these poles are also « redundant » unless $-\lambda$ is large enough to have one or more of these poles in the upper half-plane; for large enough values of $-\lambda$, we have a finite number of non-redundant poles in the upper half-plane corresponding to true bound states. In any case the s -matrix is a meromorphic function admitting a Weierstrass factorization.

4. – Potentials derived from the confluent hypergeometric equation.

We now turn to transformations of the confluent hypergeometric differential equation (7):

$$\frac{d^2 u(z)}{dz^2} + \frac{c - z}{z} \frac{du(z)}{dz} - \frac{a}{z} u(z) = 0,$$

whose general solution is denoted

$$u(z) = \varphi(a, c; z) = \lim_{b \rightarrow \infty} F\left(a, b; c; \frac{z}{b}\right),$$

which is an entire function of Z and a ; considered as a function of c it has poles at $c = -m, m = 0, 1, 2, \dots$. Transformation of (4.1) to the form (1.1) requires:

$$\left\{ \begin{aligned} z = f(r); \quad u(z) = g(r) \Phi(r); \quad h(r) = \frac{d}{dr} \{ \log_e g(r) \}, \\ \frac{d}{dr} f(r) = M g^2(r) f^c(r) \exp[-f(r)], \\ \frac{d}{dr} h(r) - h_2(r) - \frac{a}{f(r)} \left(\frac{d}{dr} f(r) \right)^2 = k^2 + V(r). \end{aligned} \right.$$

Again we consider a particular choice

$$\begin{aligned} f(r) &= \alpha r, \\ g(r) &= \frac{\sqrt{\alpha}}{M} (\alpha r)^{-c/2} \exp\left[\frac{\alpha r}{2}\right], \\ h(r) &= \frac{\alpha}{2} - \frac{c}{2r}; \quad k^2 + V(r) = -\frac{\alpha^2}{4} + \frac{\alpha}{r} \left(\frac{c}{2} - a\right) + \frac{c}{2} \left(1 - \frac{c}{2}\right) \frac{1}{r^2}. \end{aligned}$$

If we take

$$\alpha = 2ik; \quad a = \frac{c}{2} - \frac{\beta}{2ik},$$

where c, β are independent of k we satisfy the requirement that $V(r)$ is independent of k . In fact, we have

$$V(r) = \frac{\beta}{r} + \frac{c(c-2)}{4r^2}$$

For $c=0$ or $c=2$ this reduces to the Coulomb potential. Provided c is not an integer the general solution is

$$\left\{ \begin{array}{l} \Phi(r) = \frac{y_2(0)y_1(r) - y_1(0)y_2(r)}{y_2(0)y_1'(0) - y_1(0)y_2'(0)}, \\ y_1(r) = \varphi\left(\frac{c}{2} - \frac{\beta}{2ik}, c; 2ikr\right) \\ y_2(r) = (2ik)^{1-c} \varphi\left(1 - \frac{c}{2} - \frac{\beta}{2ik}, 2 - c; 2ikr\right) \end{array} \right.$$

Since this potential vanishes only as r^{-1} we do not expect to get an asymptotic form consisting only of two plane waves. To obtain the asymptotic form it is advantageous to re-express $\varphi(r)$ in terms of Whittaker functions (7):

$$\varphi(r) = AW_{\beta/2ik, (c-1)/2}(2ikr) + BW_{-\beta/2ik, (c-1)/2}(-2ikr)$$

where A and B are constants to be determined so that $\varphi(0) = 0$ and $\varphi(r)$ is continuum-normalized. For large values of $|r|$ it is known that

$$\begin{aligned} W_{\beta/2ik, (c-1)/2}(2ikr) &\sim \exp[-ikr](2ikr)^{-\beta/2ik}, \\ W_{-\beta/2ik, (c-1)/2}(-2ikr) &\sim \exp[ikr](-2ikr)^{-\beta/2ik} \end{aligned}$$

Writing $\varrho = r + (\beta/2k^2) \log r$ we may write

$$\varphi(r) \sim A(2ik) \exp[-ik\varrho] + B(-2ik) \exp[ik\varrho]$$

which has the familiar form of Coulomb scattering state solutions. In this case clearly no s -matrix in the usual sense exists!

As a second choice, consider the transformation equations (4.2) with

$$f(r) = \exp[-\alpha r],$$

$$h(r) = \frac{\alpha}{2}(c-1) \frac{\alpha}{2} \exp[\alpha r],$$

$$k^2 + V(r) = \frac{\alpha^2}{4}(c-1)^2 + \frac{\alpha^2}{2}(c-2a) \exp[-\alpha r] - \frac{\alpha^2}{4} \exp[-2\alpha r]$$

We choose the parameters

$$c = 1 + \frac{2ik}{\alpha}; \quad 2a = 1 - \frac{2ik}{\alpha} - \gamma,$$

where α, γ are independent of k . Then we get the potential

$$(4.6) \quad V(r) = \alpha \exp[-\alpha r] (\exp[-\alpha r] - 2\gamma).$$

For this potential the general solution (for c not an integer) is given by:

$$(4.7) \quad \varphi(r) = \exp\left[-\frac{1}{2} \exp[-\alpha r]\right] \cdot \left\{ A \exp[ikr] \varphi\left(\frac{1-\gamma}{2} + \frac{ik}{\alpha}, 1 + \frac{2ik}{\alpha}; \exp[-\alpha r]\right) + B \exp[-ikr] \varphi\left(\frac{1-\gamma}{2} + \frac{ik}{\alpha}, 1 + \frac{2ik}{\alpha}; \exp[-\alpha r]\right) \right\},$$

where A and B are to be determined so that $\varphi(0) = 0$ and $\varphi(r)$ is continuum-normalized. Using the fact that for $r \rightarrow \infty$, $\exp[-\alpha r] \rightarrow 0$ and $\varphi(a; c; \exp[-\alpha r]) \rightarrow 1$ we get the asymptotic form:

$$\varphi(r) \sim A \exp[ikr] + B \exp[-ikr].$$

This together with the requirement $\varphi(r) = 0$ yields the s -matrix explicitly in the form:

$$(4.8) \quad s(k) = \frac{B}{A} \frac{\varphi(1-\gamma)/2 - ik/\alpha, 1 - 2ik/\alpha; 1}{ik/\alpha, 1}.$$

No immediate statements regarding the analytic properties of $s(k)$ seem possible.

5. - Transformations of the Bessel differential equation.

As a final example of the derivation of solvable potentials we consider the Bessel differential equation

$$\frac{d^2}{dr^2}$$

Proceeding, as before, we consider the transformation of this equation to the form (1.1) by the substitutions:

$$z = f(r); \quad u(z) = g(r) \varphi(r); \quad h(r) = \frac{d}{dr} \{\log_e g(r)\},$$

$$\frac{d}{dr} f(r) = M f(r) g^2(r),$$

$$\frac{d}{dr} h(r) - h^2(r) + \left(\frac{df(r)}{dr}\right)^2 \left(1 - \frac{v^2}{f(r)}\right) = k^2 + V(r).$$

For the special case

$$f(r) = kr,$$

$$g(r) = \frac{1}{M} r,$$

we get the potential function

$$V(r) = \frac{1 - 4v^2}{4r^2}$$

$$\varphi(r) = \sqrt{r} \{A J_\nu(kr) + B J_{-\nu}(kr)\}.$$

The condition $\varphi(0) = 0$ requires $B = 0$ ($\nu > 0$); here the s -matrix is independent of k and is given by

$$s(k) = \exp[i\pi(\nu - y_2)] = -i \exp[i\pi 2].$$

Similarly for the choice

$$f(r) = \exp[-\alpha r],$$

$$g(r) = -\sqrt{\alpha}/M,$$

$$h(r) = 0,$$

we get the potential function

$$V(r) = \alpha^2 \exp[-2\alpha r]$$

The general solution is

$$\varphi(r) = A J_{ik/\alpha}(\exp[-\alpha r]) + B J_{-ik/\alpha}(\exp[-\alpha r])$$

$$A = \alpha J_{-ik/\alpha}(1) \{ J_{+ik/\alpha}(1) J'_{-ik/\alpha}(1) - J_{-ik/\alpha}(1) J'_{ik/\alpha}(1) \}^{-1},$$

$$B = \alpha J_{ik/\alpha}(1) \{ J_{ik/\alpha}(1) J'_{-ik/\alpha}(1) - J_{-ik/\alpha}(1) J'_{ik/\alpha}(1) \}^{-1}$$

This particular case has already been studied in some detail (2) and gives « redundant » poles for

$$k = ni\alpha, \quad n = 1, 2, \dots$$

and a sequence of zeroes which are the mirror images of these poles. With this repulsive potential there will be no true bound states.

6. - Concluding remarks.

In the preceding sections we have discussed a variety of potentials for which the radial Schrödinger equation for the *s*-state can be solved analytically to express the solutions in terms of « known » functions. We have also shown how to obtain the *s*-matrices explicitly in many of these cases. While the present work does not aim to give an exhaustive compilation of such potentials, it is hoped that the methods developed here will find application to wider classes of potentials.

The class of solvable potentials found in this paper can be extended significantly using Darboux's theorem (5) for second order differential equations: If the general solution $\varphi = \varphi(x)$ of the differential equation:

$$\frac{d^2\varphi}{dx^2} = \{h + a(x)\} \varphi,$$

is known for all values of *h*, then the general solution of the differential equation

$$\frac{d^2\psi}{dx^2} = \left\{ h + h_1 + f(x) \frac{d}{dx} \left(\frac{1}{f(x)} \right) \right\} \psi,$$

is given by

$$\psi(x) = f(x) \frac{d}{dx} \left(\frac{\varphi(x)}{f(x)} \right),$$

provided $h \neq h_1$ and

$$\frac{d^2 f(x)}{dx^2} = \{h_1 + a(x)\} f(x)$$

As an example of the construction of solvable Schrödinger equations of diverse types that can be obtained utilizing Darboux's theorem consider the simple solvable Schrödinger equation with the exponential potential (2):

$$\frac{d^2 \varphi}{dr^2} + (k^2 + \alpha^2 \exp[-2\alpha r]) \varphi = 0$$

The general solution is

$$\varphi(r) = A J_{ik}(\exp[-\alpha r]) + B J_{-ik}(\exp[-\alpha r])$$

and a particular solution corresponding to $k = k_1 = \alpha/4$ is given by

$$f(r) = \exp[-\frac{1}{2}\alpha r] \sin(\exp[-\alpha r])$$

This choice leads to a solvable Schrödinger equation with the potential

$$V(r) = \text{const} + f(r) \frac{d^2}{dr^2} \left(\frac{1}{f(r)} \right) =$$

The number of such examples can be multiplied indefinitely.

The analytic structure of these s -matrices are of some interest in themselves, particularly the s -matrix discussed in Section 3. The sequence of poles associated with these potentials is a reflection of the nature of the basic differential equations selected for study; to obtain a more complicated analytic structure (say, branch points), for the s -matrix we must select different types of differential equations.

In these transformations we have required that the Schrödinger equations must contain only velocity-independent potentials. By relaxing this requirement a very wide variety of velocity-dependent potentials with associated meromorphic s -matrices could be constructed; the lack of velocity-dependence has been a hindrance rather than a help in our investigation. Nevertheless we have been conservative and restricted ourselves to conventional velocity-independent potentials.

It may not be uninteresting to point out here that the phase-shift or the s -matrix for an arbitrary Hermitian potential need not have any analytic properties whatever. The simplest example is given by the (nonlocal) separable potential whose expression in momentum space is of the form $V(k, k') =$

$= \lambda f(k^2) f^*(k'^2)$. The s -matrix for real momenta can be calculated to obtain:

$$s(k) = \frac{\alpha(k^2 + i\varepsilon)}{\alpha(k^2 - i\varepsilon)},$$

with

$$\alpha(z) = 1 - \lambda \int_0^{\infty} \frac{f(\xi) f^*(\xi)}{\xi - z} d\xi$$

In the general case of an arbitrary function of the real variable, k^2 not having any specific analytic properties the potential is still Hermitian but $s(k)$ has no analytic properties but is unimodular in the interval $0 < k < \infty$. In view of this circumstance the study of analytic velocity-dependent potentials may be of intrinsic interest.

RIASSUNTO (*)

Si presenta un metodo sistematico per costruire potenziali (indipendenti dalla velocità) per cui l'equazione di Schrödinger dell'onda s può essere risolta in termini di funzioni note. Si costruiscono molti esempi di questo tipo e si precisano le strutture analitiche di alcune delle corrispondenti ampiezze di scattering. L'uso del teorema di Darboux permette una significativa estensione della classe dei potenziali risolvibili.

(*) Traduzione a cura della Redazione.