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Relativistic Invariance and Hamiltonian Theories of Interacting Particles*

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I. INTRODUCTION

We distinguish two different kinds of assumptions that occur under the heading of "relativistic invariance" in theories constructed to give a relativistically invariant description of interactions between particles. The first of these reflects the principle of special relativity that the laws of physics should be invariant under changes of reference frame; it is formulated in terms of the symmetry of the theory under the group of frame transformations. The second is an assumption of "manifest invariance"; it requires that certain quantities transform under changes of reference frame in a particular manner that is intimately related to the Lorentz (or Galilei) transformations of space–time events. The requirement of symmetry under the relativistic transformation group can be satisfied in a very general and simple manner through the construction of quantities satisfying the Lie bracket equations characteristic of generators of the group. A requirement of "manifest invariance" by itself also may be satisfied quite generally and simply. But the combined requirements of relativistic symmetry and manifest invariance may restrict the theory so severely that it is capable only of describing noninteracting particles. We will show that this is in fact the case in a Lorentz symmetric classical mechanical theory of the motion of a pair of particles. The positions of the particles as a function of time trace out the world lines of the two particles. If the coordinates of the space–time events comprising these world lines are required to transform in the usual manner of Einstein–Lorentz, it is found that each particle must move with a constant velocity.

The idea that relativistic symmetry can be built into a theory through the construction of a realization of the relativistic transformation group is familiar in quantum mechanics and particularly in relativistic quantum field theory where the unitary representations of the Lorentz group have played a central role.[1–3] The essential feature is that there is a unitary operator which transforms a description of the system with respect to a given reference frame to the description with respect to each relativistically equivalent frame, and that the set of such unitary operators constitutes a representation of the group of transformations of reference frames. That the theoretical structures available for describing the system are the same for two different reference frames is assured by the unitary relation between them. That the rule for transforming a description from one frame to another is itself invariant under changes of reference frame is assured by the group structure. Since the dynamical transformations that relate descriptions of the system at different instants of time are identified with transformations to reference frames displaced in time, the invariance of the dynamical laws under changes of reference frame follows from the group structure as a particular case of the above. The relativistic symmetry is thus introduced by the construction of a representation of the Lorentz group. If this representation is continuous it is sufficient for the establishment of relativistic symmetry to construct the ten operators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ which are the Hamiltonian (generator of time translations) and the generators of space-translations, space rotations, and pure Lorentz transformations, respectively, and which satisfy the commutation relations characteristic of generators of the Lorentz group.

This method of introducing relativistic symmetry by constructing the generators of the relativistic transformation group is a very general one and is by no means bound either to field theory or to quantum mechanics. Dirac has pointed out that this method could be used to construct a relativistically symmetric theory of the interaction of a fixed number of particles in terms of functions of their canonical variables.[4] Thomas, Bakamjian, and Foldy have

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1 E. P. Wigner, Ann. Math. 40, 149 (1939).
2 A. S. Wightman, Phys. Rev. 101, 860 (1956).
3 A. S. Wightman, Nuovo Cimento, Suppl. 14, 81 (1959).
4 P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).

shown how to construct the ten generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ for such a theory which include, in the structure of H, any of a large class of interactions.[5,6] That the requirement of Lorentz symmetry in a particle theory which includes interaction can be satisfied through the construction of a set of generators has also been emphasized by one of the authors.[7] These theories can be constructed in classical mechanics by requiring the generators of canonical transformations to satisfy Poisson bracket equations characteristic of the group structure, as well as in quantum mechanics where the generators of unitary transformations are required to satisfy characteristic commutation relations. The theory can be made symmetric under either the Galilei or the Lorentz group by requiring the generators to satisfy the respective Lie bracket equations characteristic of that group.

In the following section of this paper we discuss relativistic symmetry, or symmetry under a relativistic transformation group, in terms of a Lie group formalism which abstracts the relevant features common to both classical and quantum mechanics whether of particles or of fields. In terms of this formalism we show just what we mean when we say that relativistic symmetry is established in the construction of the ten generators satisfying the Lie bracket equations characteristic of the relativistic transformation group.

A theory exhibiting relativistic symmetry in the ten generators of the Lorentz (or Galilei) group may still require a further assumption about the specific manner in which certain quantities of the theory transform. We may require, for example, that certain quantities transform as tensors of a given rank so that equations can be written in a manifestly covariant form. In relativistic quantum field theory it is assumed that the basic quantities transform under the unitary representation of the Lorentz group as scalar, spinor, or vector fields. The importance of this assumption in field theory can be seen in the role it plays in the establishment of results such as the *TCP* theorem, the connection between spin and statistics, and the substitution law. In this paper, we consider a classical mechanical theory in which the motion of a fixed number of particles is described by the time dependence of the positions of the particles in space, and we assume the manifest invariance property that the world line of a particle transforms as a sequence of space–time events according to the usual Lorentz transformation formula.

In the formalism used in this paper the transformations of a quantity under changes of reference frame are determined by the Lie brackets of that quantity with the ten generators of the relativistic transformation group. A manifest invariance assumption can therefore be formulated in terms of a set of equations involving these Lie brackets. We show in Sec. III, for a Lorentz symmetric classical mechanical theory describing the motion of a fixed number of particles, that if the coordinates of the space–time events determined by the positions of a particle as a function of time transform in the familiar manner according to the Lorentz transformation formula under the Lorentz group of transformations of reference frame, then the quantities Q_i, $i = 1,2,3$, representing the position of the particle satisfy the equations

$$[Q_i,P_j] = \delta_{ij}$$
$$[Q_i,J_j] = \epsilon_{ijk}Q_k$$
$$[Q_i,K_j] = Q_j[Q_i,H]$$

with the generators of the Lorentz group. (In the Galilean case equations are obtained that are identical, except that the right-hand side of the last equation is zero.) One of the authors has independently established essentially these same equations by a method which uses only the world line of the particle and the geometric properties of the Lorentz transformations in space–time and which is independent of the formalism used here.[8]

Thus, in a classical mechanical Lorentz-invariant theory of particle motion there are two requirements to be satisfied and these are formulated in two sets of equations—the Poisson bracket equations satisfied by the generators exhibiting the symmetry of the theory under the Lorentz group and the above equations which specify the explicit transformation properties of the particle positions Q_i. Of the latter equations the first two are the more familiar ones and are usually assumed to be valid in that the generators $\mathbf{P}$ and $\mathbf{J}$ are assumed to have the standard form for noninteracting particles.[4–6] (The usual description of a fixed number of noninteracting spinless particles is outlined in Appendix A.)

[5] L. H. Thomas, Phys. Rev. **85**, 868 (1952); B. Bakamjian and L. H. Thomas, *ibid.* **92**, 1300 (1953); B. Bakamjian, *ibid.* **121**, 1849 (1961).

[6] L. L. Foldy, Phys. Rev. **122**, 275 (1961). We recommend that the reader who is unfamiliar with this point of view read, in particular, the introduction to this paper.

[7] E. C. G. Sudarshan, *1961 Brandeis Summer Institute Lectures in Theoretical Physics* (W. A. Benjamin Company, New York, 1962), and "Hamiltonian Dynamics of Relativistic Particles" (University of Rochester report NYO—9680, unpublished).

[8] D. G. Currie, *Interaction contra Classical Relativistic Hamiltonian Particle Mechanics* (University of Rochester report NYO-10242, to be published), and "The Hamiltonian Description of Interaction for Classical Relativistic Particles" (Thesis, University of Rochester, 1962, unpublished).

finite-dimensional subspace of R which is a Lie subalgebra of R; this means that the Lie bracket $[A,B]$ of any two elements A,B of L belongs to L. We represent transformations of reference frames by linear (inner) automorphisms of R generated by the elements of L.

Let H be an element of L. We denote by $e^{[H]t}$, t real, the automorphisms of R belonging to the one-parameter group generated by H. We can think of these as being defined by

$$e^{[H]t}(A) = A + [A,H]t + \tfrac{1}{2}[[A,H],H]t^2 + \cdots \quad (2.4)$$

for each A in R for those values of t for which the series is meaningful. Or we can think of $e^{[H]t}(A)$ as being a family of elements of R depending on the parameter t in such a way that they satisfy the first-order differential equation

$$(\partial/\partial t)e^{[H]t}(A) = [e^{[H]t}(A),H] \quad (2.5)$$

with the boundary condition

$$e^{[H]t}(A)|_{t=0} = A\,. \quad (2.6)$$

As H varies over the Lie algebra L, the automorphisms $e^{[H]t}$ generate a Lie group G which has L as its Lie algebra.[13] (See further discussion of the definition of G from L in Appendix B.)

In the Lie group formalism of classical mechanics the Lie bracket is the Poisson bracket, L is a finite-dimensional subspace of real functions on phase space which is closed under the operation of taking Poisson brackets, and G is the group of canonical transformations for which the elements of L are infinitesimal generators. For each real function A belonging to R, $e^{[H]t}(A)$ is a function of the phase-space coordinates and of the parameter t that satisfies the differential equation (2.5) (where the bracket is now a Poisson bracket) with the boundary condition (2.6).[14] In the Lie group formalism of quantum mechanics the Lie bracket is the commutator divided by $i\hbar$, L is a finite-dimensional subspace of self-adjoint operators which is closed under the operation of commutation, and G is the group of unitary transformations for which the elements of L are infinitesimal generators.[15] For each operator A belonging to R, $e^{[H]t}(A)$ is the operator $U_t^\dagger A U_t$, where

$$U_t = e^{-iHt}$$

is the one-parameter group of unitary operators having H as infinitesimal generator.[16–18]

We make two further postulates about the mathematical structure of our formalism: that the subset S of R is invariant under the group G of automorphisms, and that the bilinear functional (A,F) is unchanged when the same automorphism in G is applied to both A and F. For any H belonging to L and any real t, we have that $e^{[H]t}(F)$ belongs to S if F belongs to S, and

$$(e^{[H]t}(A),e^{[H]t}(F)) = (A,F) \quad (2.7)$$

for any A in R and F in S. The Lie group formalism of classical mechanics satisfies these postulates in that each canonical transformation is equivalent to a measure-preserving transformation of phase space, so that the image under a canonical transformation of a positive definite function on phase space is a positive definite function, the integral over phase space of a function is equal to the integral of the canonically transformed function, and the image of a product of functions is equal to the product of the images of the functions. Canonical transformations map density functions to density functions and preserve the integral over phase space of a product of functions. In quantum mechanics these properties are consequences of the fact that a unitary transformation preserves the self-adjoint property and positive definiteness of operators, the trace of an operator, and multiplication of operators. Unitary transformations take density operators to density operators and leave the trace of a product of operators invariant.[19] From the form taken in classical mechanics, we can see that the invariance of the subset S and of the bilinear functional (A,F) is a generalization of Liouville's theorem.[14] This will become more apparent as

[13] For the general theory of Lie Groups and Lie algebras see, for example, L. Pontrjagin, *Topological Groups*, translated by E. Lehmer (Princeton University Press, Princeton, New Jersey, 1939); G. Racah, Nuovo Cimento, Suppl. **14**, 67 (1959); or P. M. Cohn, *Lie Groups* (Cambridge University Press, New York, 1957).

[14] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1950); D. ter Haar, *Elements of Statistical Mechanics* (Rinehart and Company, New York, 1954).

[15] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, New York, 1958), Chaps. IV, V.

[16] The rigorous connection between the one-parameter groups of unitary operators and the self-adjoint operators is given by Stone's theorem. See, for example, F. Riesz and B. sz-Nagy, *Functional Analysis*, translated by L. F. Boron (Frederick Ungar Publishing Company, New York, 1955), pp. 380–385.

[17] One can also formulate quantum mechanics in terms of functions on phase space in which case the Lie bracket is the Moyal bracket. J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949).

[18] For an outline and comparison of classical mechanics and quantum mechanics in the Lie group dynamical formalism and for a list of further references, see T. F. Jordan and E. C. G. Sudarshan, Rev. Mod. Phys. **33**, 515 (1961).

[19] The one-parameter groups of unitary transformations are, in fact, the only one-parameter groups of linear automorphisms of the operator space that leave the subset of density operators invariant. T.F. Jordan, M. A. Pinsky, and E. C. G. Sudarshan, J. Math. Phys. **3**, 848 (1962).

we see how this invariance allows us to relate transformations of the quantities A in R to transformations of the representatives F in S of states.

Let H be an element of L. We may think of H as the Hamiltonian or generator of time translations. Suppose that A in R represents a particular physical quantity (for example, the position of a particle), and that F in S represents a particular state of the system. We may think of these as being part of the description of the system at time zero with respect to a given reference frame, so that the expectation value of the quantity represented by A is (A,F). We generate a description of the system at time zero with respect to a second reference frame translated an amount t in time by letting F represent the state of the system in the second description as well as in the first, and by letting $e^{[H]t}(A)$ represent, in the second description, the quantity that was represented by A in the first description. The expectation value of this quantity, in the second description, is then

$$(e^{[H]t}(A),F)\,. \tag{2.8}$$

This procedure for relating the descriptions of the system with respect to different time coordinate frames may be called the "Heisenberg picture" of time translations. An alternative procedure is provided by the "Schrödinger picture" in which we let $e^{[H](-t)}(F)$ represent, with respect to the second frame, the state which was represented by F with respect to the first frame, and let A represent the same quantity with respect to both frames. The expectation value of this quantity with respect to the second frame is

$$(A,e^{[H](-t)}(F)) \tag{2.9}$$

which is equal, by virtue of the invariance property (2.7) of the bilinear functional, to the expectation value (2.8) obtained in the Heisenberg picture. The Heisenberg and Schrödinger pictures are equivalent ways of representing transformations of time coordinate frames. In either case the change of the expectation values of physical quantities, which is all that is physically important, is the same.

For a given description of the physical system at time zero with respect to a given reference frame, we have a procedure for constructing the description of the system at the zero in time with respect to a frame of reference displaced in time from the frame with respect to which our original description was given. In general, we want to be able to translate the given description to a description with respect to any relativistically equivalent frame of reference; that is, any frame which may be displaced in space, rotated in space, moving with a uniform velocity, or displaced in time with respect to the given frame. Besides the Hamiltonian H which generates translations in time, we want the Lie algebra L to contain a triplet of elements $\mathbf{P}$ generating translations in space, a triplet of elements $\mathbf{J}$ generating rotations in space, and a triplet of elements $\mathbf{K}$ generating transformations to uniformly moving frames. We assume that L consists of linear combinations of these ten elements plus possibly "neutral" elements; that is, elements which have vanishing Lie brackets with every element of R (see Appendix B). To assure that L is a Lie algebra it is necessary and sufficient to assume that the Lie bracket of any two of these ten elements belongs to L or is a linear combination of these ten elements plus possibly a neutral element. The specification of these Lie bracket relations completely determines the structure of the Lie algebra L and the Lie group G of automorphisms generated by L.

We are interested in the two cases where G is a realization of the covering group of the inhomogeneous proper Lorentz or Galilei group. The structure of G for these cases is determined in our formalism by the requirement that L be a realization (up to neutral elements) of the Lie algebra of the Lorentz or Galilei group, respectively—i.e., the Lie bracket relations for the ten elements H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ must be those of generators of the Lorentz or Galilei group. In either case we must have

$$\begin{gathered}[P_i,P_j]=0\,,\quad [P_i,H]=0\,,\quad [J_i,H]=0\\ [J_i,J_j]=\epsilon_{ijk}J_k\,,\quad [J_i,P_j]=\epsilon_{ijk}P_k\\ [J_i,K_j]=\epsilon_{ijk}K_k\,,\quad [K_j,H]=P_j\end{gathered} \tag{2.10}$$

for $i, j, k = 1, 2, 3$ (always in this paper $\hbar = c = 1$). In the case of the Lorentz group we must have in addition that

$$[K_i,K_j]=-\epsilon_{ijk}J_k\,,\quad [K_i,P_j]=\delta_{ij}H\,, \tag{2.11}$$

while for the Galilei group we must have that

$$[K_i,K_j]=0\,,\quad [K_i,P_j]=\delta_{ij}M\,. \tag{2.12}$$

To insure a representation of the Lorentz or Galilei group these equations actually need to be satisfied only to within the addition of neutral elements to the right-hand sides, since these added neutral elements will not change the structure of the group G of automorphisms (see Appendix B). Hence, the "constant" mass M which is the "nonrelativistic limit" of H appearing in the second of Eqs. (2.12) is not a linear combination of the ten generators but is a neutral element.[6] One can show that by adding appropriate neutral elements to the ten generators

H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$—which does not change the automorphisms they generate—one can eliminate all of the neutral elements that may occur in the Lie bracket equations for the Lorentz group and in the case of the Galilei group one can eliminate all of the neutral elements except the M that occurs in the second of Eqs. (2.12) (see Appendix B). Therefore, we will work with Eqs. (2.10) and (2.11) or (2.12), as given, to characterize the Lorentz or Galilei group, respectively. For an example of generators satisfying these equations the reader is referred to the description of noninteracting particles in Appendix A.

The point we want to make is that we can construct dynamical theories which have the structure we have outlined, including the realization G of the Lorentz or Galilei group generated by ten elements of L satisfying Eqs. (2.10) and (2.11) or (2.12). These theories satisfy the principle of Einstein–Lorentz or Galilean relativity or symmetry under the Lorentz or Galilei group, respectively. To show exactly what is meant by this statement we will use the formalism we have introduced to outline the following: the equivalence of instantaneous descriptions of the system with respect to different relativistically equivalent reference frames; the introduction of dynamics or time dependence of descriptions; the invariance of the dynamical laws or equations of motion under transformations of reference frames.

We have already seen how a description of the system at time zero with respect to a given frame of reference can be transformed, by use of the one-parameter group of automorphisms of R generated by the Hamiltonian H, into a description at time zero with respect to a frame of reference displaced in time from the given frame. We had a choice of using either the Heisenberg or Schrödinger picture. Now we want to use the same kind of transformation, generated by $\mathbf{P}$, $\mathbf{J}$, or $\mathbf{K}$, to construct a description of the system with respect to a frame of reference displaced in space, rotated in space, or moving uniformly with respect to the given frame. For the moment we use only the Heisenberg picture.

Let T be an element of L. In particular, we can think of T as any one of the ten generators H, $\mathbf{P}$, $\mathbf{J}$, or $\mathbf{K}$. Suppose that in the given description, at time zero with respect to a given frame of reference, F represents the state of the system and A represents some physical quantity. We construct a description with respect to a new frame of reference by letting F represent the state of the system and letting $e^{[T]s}(A)$ represent the physical quantity at time zero with respect to the new frame. If T is equal to H, P_j, J_j, or K_j then the new frame is translated by an amount s in time, displaced by s in space in the j direction, rotated in space by an angle s about the j axis, or moving uniformly in the j direction with a velocity s (Galilei) or tanh s (Lorentz) with respect to the given frame, respectively. The quantity represented by A with respect to the given frame has the expectation value (A,F) in the given description, while it has the expectation value $(e^{[T]s}(A), F)$ in the description with respect to the new frame.

This construction gives us descriptions of the same physical system (in the same physical state) with respect to different frames of reference. The expectation values occuring in these descriptions represent measurements made on the same physical system with respect to the different frames of reference and therefore are not, in general, the same in each description. However, the form or structure of the description is the same for every relativistically equivalent frame. Although a specific physical state of the system looks different from each different frame, the theory does not distinguish the description with respect to any one frame as being different from any other. For every element A of R, representing some quantity in the description with respect to the given frame, there is an element $e^{[T]s}(A)$ of R representing the quantity in the description with respect to the new frame. For any element F of S representing a state of the system and giving the expectation value (A,F) of that quantity in the former description, there is an element $e^{[T]s}(F)$ of S which represents a possible state of the system and gives the same expectation value

$$(e^{[T]s}(A), e^{[T]s}(F)) = (A,F)$$

of that quantity in the latter description. With respect to each frame of reference we can describe the system in terms of the same set of quantities and states or measurements giving the same set of possible results.

So far we have only considered the transformation of a description of a physical system at a given time with respect to a given initial frame of reference to a description with respect to a second frame of reference. The group structure of our formalism enters when we ask if the rule for transforming the second description to a description with respect to a third frame is the same as that which we used to get the second from the first—that is, if the description thus obtained with respect to the third frame is the same as we would get by transforming the initial description directly from the first to the third frame. To satisfy the principle of relativity we not only need to have a description with respect to each equivalent frame identical in form to the given description with respect to the initial frame, but we also must be able to transform *from* the description with respect to each second frame to a description with respect to any other relativistically equivalent frame in a manner identical to that used in transforming *from* the description with respect to the initial frame.

The fact that the group G of automorphisms of R is a realization of the group (Lorentz or Galilei) of transformations of frames assures that the successive transformations of a description from an initial to a second and then from the second to a third frame yield the same result as a transformation from the initial to the third frame. Suppose that the transformation of the description from the initial to the second frame is realized by the automorphism $e^{[T]s}$ of the elements A of R; this transformation is of amount s and of the kind generated by T with respect to the initial frame; for example, if T is J_3 the second frame is rotated about the z axis through an angle s with respect to the initial frame. Suppose that the transformation from the second to third frame is of amount r and of the kind which would be generated by the element V of L if this transformation were made with respect to the *initial* frame; for example, if V is P_1 then the third frame is displaced a distance r in the x direction with respect to the *second* frame. The representative of this generator in the description with respect to the second frame is $e^{[T]s}(V)$, so that the automorphism which realizes the transformation of the description from the second to third frame is

$$e^{[\exp([T]s)(V)]r}\,.$$

The image of a quantity A under the combined automorphism which realizes the transformation of the description from the first to the third frame is then

$$e^{[\exp([T]s)(V)]r}(e^{[T]s}(A))\,.$$

In our example this is

$$e^{[\exp([J_3]s)(P_1)]r}(e^{[J_3]s}(A)) = e^{[P_1 \cos s+P_2 \sin s]r}(e^{[J_3]s}(A))\,,$$

where we see that the rotation by the angle s about the z axis with respect to the first frame is followed by displacement by $r \cos s$ in the x direction and $r \sin s$ in the y direction with respect to the first frame, corresponding to displacement by r in the x direction with respect to the second frame.

Now we can use the identity (see Appendix C)

$$e^{[\exp([T]s)(V)]r}(e^{[T]s}(A)) = e^{[T]s}(e^{[V]r}(A)) \quad (2.13)$$

to see in another way the relation between a transformation of a description from the second frame and a similar transformation of the description from the initial frame. This equation tells us that as an alternative to transforming the description from the first to the second and then from the second to the third frame (left-hand side) we may equivalently transform the description from the initial frame to a fourth frame related to the initial frame in the same way that the third frame is related to the second, and then transform, still with respect to the first frame, to the description with respect to the second frame (right-hand side). In other words, the transformation between the descriptions with respect to the second and third frames is the image under the transformation from the initial description to the second description of the transformation between the descriptions with respect to the equivalently related first and fourth frames. Still another way of seeing this is to rewrite the identity (2.13) in the form

$$e^{[\exp([T]s)(V)]r}(A) = e^{[T]s}(e^{[V]r}(e^{[T](-s)}(A)))\,; \quad (2.14)$$

which shows that to obtain the image in the description with respect to the third frame of an element A in the description with respect to the second frame we can transform back to the initial description, do the equivalent transformation with respect to the first frame, and then again transform to the second description.

As yet, we have considered only descriptions of the system at a single instant of time and transformations of such descriptions between relativistically equivalent reference frames. To have a dynamical theory we must have a law which states how the description of the system changes in time; in a relativistically invariant dynamical theory this dynamical law must be invariant under changes of reference frame. We introduce dynamics in our formalism in a familiar and simple way. We postulate that the same transformation that changes a given description of the system at time zero with respect to a given frame to the description at time zero with respect to a second frame displaced in time by an amount t also changes the given description to the description at time t with respect to the same frame. In other words, the description at time t with respect to the given frame is the same as the description at time zero with respect to a frame displaced in time by an amount t. If, at time zero with respect to the given frame, F in S represents the state of the system and A in R represents a particular quantity, then F and $e^{[H]t}(A)$ represent the state and that quantity both at time t with respect to the given frame and at time zero with respect to the frame displaced in time by an amount t. This is the "Heisenberg picture" of dynamical transformations. The equivalent "Schrödinger picture" is similarly defined in the obvious way.

That the dynamical law which we have introduced is invariant under transformations of reference frames follows as a particular case of the invariance of the rule for transforming a description from one frame to another. This invariance was shown above to result from the group property of the automorphisms that realize these transformations. To explicitly exhibit the invariance of the dynamics we consider the particular case of the above discussion in which V is equal to the Hamiltonian H and T is equal to some other generator of frame transformations; for example, the generator K_1 of transformations to a frame moving uniformly in the x direction. In other words, the second frame of the above discussion is moving with a uniform velocity s (or tanh s) in the x direction with respect to the initial frame, and the third frame is displaced by r in time with respect to the second frame. Alternatively, we can think of the displacement in time as giving a description at a different time with respect to the same frame. Let t and t' denote the time coordinates of the first and second frames. Then, if A represents a particular quantity at $t = 0$ with respect to the first frame,

$$e^{[H]r}(A) \qquad (2.15)$$

represents that quantity at time $t = r$ with respect to the first frame and

$$e^{[\exp([K_1]s)(H)]r}(e^{[K_1]s}(A)) \qquad (2.16)$$

represents that quantity at time $t' = r$ with respect to the second frame. The images (2.15) of the quantities A give a description of the system at time $t = r$, and as r takes on all real values they give a "complete dynamical description," that is, a description for every time, with respect to the first frame. Similarly the quantities (2.16), for all values of $t' = r$, give a "complete dynamical description" with respect to the second frame. Now, if we let $e^{[K_1]s}(H) = H'$, $e^{[K_1]s}(A) = A'$ (in general, a primed element is the image in the second description of an element in the first description), the identity (2.13) is

$$e^{[H']r}(A') = (e^{[H]r}(A))', \qquad (2.17)$$

which shows how a "complete dynamical description" with respect to the first frame is transformed into a "complete dynamical description" with respect to the second frame. We get the same result if we transform a description from the first to the second frame at $t = t' = 0$ and then to $t' = r$ with respect to the second frame, or if we transform from $t = 0$ to $t = r$ with respect to the first frame and then transform to the description with respect to the second frame. It makes no difference at what time we choose to make the transformation between descriptions.

If we consider r to be infinitesimal, expand both sides of the identity (2.13) using power series of the form (2.4) in r, and equate the terms of the first order (see also Appendix C), we get

$$[e^{[T]s}(A), e^{[T]s}(V)] = e^{[T]s}([A,V]), \qquad (2.18)$$

which expresses the preservation of the Lie bracket by the automorphisms of G. In our particular case Eq. (2.18) is

$$[A',H'] = ([A,H])',$$

which shows how the dependence of the quantities (2.15) on $t = r$ given by the "equation of motion"

$$(\partial/\partial r)e^{[H]r}(A) = [e^{[H]r}(A),H]$$

with respect to the first frame is transformed into the dependence of the quantities (2.16) on $t' = r$ given by the "equation of motion"

$$(\partial/\partial r)e^{[H']r}(A') = [e^{[H']r}(A'),H']$$

with respect to the second frame.

We can see that a theory constructed according to such a general formalism satisfies the properties of relativistic symmetry that[20]:

a. Given a "complete dynamical description" with respect to a given reference frame, a "complete dynamical description" can be constructed with respect to any other relativistically equivalent frame.

b. If the "complete dynamical description" with respect to one frame satisfies the correct "equation of motion," the "complete dynamical description" with respect to the other frame also satisfies the correct "equation of motion."

c. The "equation of motion has the same form in both frames.

If we wish we may, of course, translate this whole discussion of relativistic symmetry into the Schrödinger picture. Previously, in the Heisenberg picture, we let the same element F of S represent the same state of the system in every description. We transformed the element A which represented a quantity in the description with respect to an initial frame to $e^{[T]s}(A)$ in the description with respect to a second frame and to the element (2.13) in the description with respect to a third frame. Now, in the Schröd-

[20] E. P. Wigner, Nuovo Cimento 3, 517 (1956).

inger picture, we let the same element A represent the same quantity in every description. But, we transform the element F of S which represents the state of the system in the description with respect to the first frame to $e^{[T](-s)}(F)$ in the description with respect to the second frame, and to

$$e^{[V](-r)}(e^{[T](-s)}(F)) = e^{[\exp([V](-r))(T)](-s)}(e^{[V](-r)}(F)) \tag{2.19}$$

in the description with respect to the third frame. [Here we have used the identity (2.13).] The equivalence of the two pictures for the second and third frames follows from the invariance property of the bilinear functional that

$$(e^{[T]s}(A),F) = (A,e^{[T](-s)}(F))$$

and

$$(e^{[T]s}(e^{[V]r}(A)),F) = (A,e^{[V](-r)}(e^{[T](-s)}(F)))\,.$$

From the left-hand side of Eq. (2.19) we see that after having transformed from the first to the second frame with the automorphism generated by T, we transform from the second to the third frame with the automorphism generated by V—which is the same element of L that we would use to generate a similar transformation from the initial frame. The generators are not transformed in the Schrödinger picture as they are in the Heisenberg picture [compare with the left-hand side of Eq. (2.13)]. On the other hand, from the right-hand side of Eq. (2.19) we see that if we first make the transformation generated by V with respect to the first frame, the generator of the transformation to the second description is changed from T to $e^{[V](-r)}(T)$ [compare with the right-hand side of Eq. (2.13)]. Let us illustrate this with the case where $V = H$ and $T = K_1$. If the state of the system is represented by F at $t = 0$ with respect to the first frame, then it is represented by

$$e^{[H](-r)}(F)$$

at $t = r$ with respect to the first frame, by $F' = e^{[K_1](-s)}(F)$ at $t' = 0$ with respect to the second frame, and by

$$\begin{aligned} e^{[H](-r)}(F') &= e^{[H](-r)}(e^{[K_1](-s)}(F)) \\ &= e^{[\exp([H](-r))(K1)](-s)}(e^{[H](-r)}(F)) \end{aligned}$$

at $t' = r$ with respect to the second frame. The Hamiltonian is the same in both descriptions. But the generator of the transformation to the second description depends on time. For either the Lorentz or Galilei group it follows from the last of Eqs. (2.10) that it is equal to

$$e^{[H](-r)}(K_1) = K_1 - rP_1$$

at time $t = r$. If we had used a component of **P** or **J** instead of **K** we would, of course, have found the generator of the transformation to the description with respect to the second frame to be time independent, since **P** and **J** have vanishing Lie brackets with H for both the Lorentz and Galilei groups [see Eqs. (2.10)]. In other words, **P** and **J** are "constants of motion" but **K** is not.

The main fact that we want to exhibit is that relativistic symmetry, or symmetry under the group of transformations of reference frames, can be achieved in a standard simple manner in any theory, constructed in accordance with our general formalism, which contains a realization G of the group of frame transformations generated by a Lie algebra L. The description of the system has an identical form with respect to each of a set of equivalent reference frames and the rule for transforming a description from one frame to another is the same for any two similarly related frames. This symmetry property can be established for any group of reference frame transformations. We need not specify what the actual structure of the group is, for example, whether it is the Lorentz or Galilei group. The requirement that G be a realization of a particular group such as the Lorentz or Galilei group represents an additional independent postulate. This postulate is contained in the requirement that the ten generators H, **P**, **J**, and **K** satisfy a particular set of Lie bracket relations such as those of Eqs. (2.10) and (2.11) or (2.12) for the Lorentz or Galilei groups.

Within our general formalism, the problem of constructing a relativistically symmetric theory describing some physical system is essentially the problem of finding suitable generators H, **P**, **J**, and **K** satisfying the correct Lie bracket equations. The choice of the generators depends on the physical identification of the elements of R and reflects the structure of the physical system. The simplest structure for a realization G of the relativistic transformation group, or for the realization L of the Lie algebra of that group, is that of the "irreducible" realizations which can not be broken down into composites of more simple realizations. In quantum mechanics all of the irreducible representations of the Lorentz group are known.[1] They can be listed according to values of a mass and a spin parameter which are the characteristic invariant quantities. The irreducible representations for which these parameters have reasonably familiar values are used to represent "elementary" physical systems such as "elementary" particles. The irreducible representations of the Galilei group are also known.[21] In classical mechanics there has been no complete investigation of the irreducible realizations of the Lorentz group by canonical transformations, although the irreducible realizations corresponding to certain mass and spin values can be constructed in analogy to the quantum mechanical representations. A systematic study of the irreducible representations of the Galilei group in classical mechanics has been made by Loinger.[22]

It is the reducible realizations that are relevant for descriptions of systems of more than one particle and it is these which interest us in this paper. The simplest of these are direct products of irreducible realizations and are taken to represent systems of noninteracting particles. (The generators for such realizations, corresponding to a fixed number of spinless noninteracting particles, are given in Appendix A.) Each of the generators is the sum of the corresponding generators for the "single-particle" realizations. In particular, the generators H, $\mathbf{P}$, and $\mathbf{J}$ are identified with the total energy, total momentum, and total angular momentum of the system, which are equal to the sums of the individual particle energies, momenta, and angular momenta, respectively.

To describe a system of interacting particles it is customary to write the Hamiltonian H as the sum of individual "free-particle" Hamiltonians plus an "interaction term" and to let the generators $\mathbf{P}$ and $\mathbf{J}$ be equal (just as in the noninteracting case) to simply the sum of the individual "free-particle" momenta and angular momenta. "Interaction" is customarily introduced by changing the total energy but not the total momentum or angular momentum. In the case of the Galilei group, one can introduce an interaction term into H while the generators $\mathbf{K}$ are also kept in the noninteracting form of the sum of the individual "free-particle" generators.[6] But in the Lorentz case, Eqs. (2.11) make this impossible.

In the remainder of this paper we give more detailed consideration to theories describing a fixed number N of particles. We consider the quantities descriptive of the system to be real functions of the $3N$ pairs of canonical variables for the N particles. (We could, of course, also include spin variables for each of the particles by assigning to them the correct Lie bracket relations, but for the sake of simplicity we limit our study to the case of spinless particles.) In a classical mechanical theory the linear space R consists of real functions defined on the $6N$ dimensional phase space, while in a quantum mechanical theory R consists of the self-adjoint operators associated with the von Neumann algebra generated by the $3N$ pairs of canonical operators. For either kind of theory it has been demonstrated that one can construct within R ten generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ satisfying the Lie bracket relations of the Lorentz or Galilei group [Eqs. (2.10) and (2.11) or (2.12)] and including, in the structure of H, any of a large class of interactions.[5–7] In this way, one can obtain a theory of interacting particles that satisfies the principle of relativistic symmetry, or symmetry under the relativistic transformation group, as discussed in this section.

The investigation of such theories has not as yet included a complete consideration of the transformation properties of the particle positions. It is to this question that we now turn.

III. RELATIVISTIC TRANSFORMATIONS OF PARTICLE POSITIONS

Our main interest in this section is to exhibit equations which characterize the transformation properties of the position of a particle in a relativistic theory and which, therefore, may be added to the Lie bracket relations (2.10) and (2.11) or (2.12) in forming the requirements a theory must fulfill in describing relativistically invariant motion of particles. For this purpose we deal specifically with classical mechanics and we consider the motion of particles as being described by the time dependence of their positions in space.

Each physical quantity is represented by a real function

$$A(q_1^1, \cdots q_3^N, p_1^1, \cdots p_3^N)$$

of the $6N$ real variables q_i^n, p_i^n, $n = 1, 2 \cdots N$; $i = 1, 2, 3$. We denote the point $(q_1^1, \cdots q_3^N,\ p_1^1, \cdots p_3^N)$ of phase space simply by (q,p) and write $A(q,p)$ for the function defined on phase space. Each state of the system is represented by a real positive-definite probability distribution function $F(q,p)$ on phase space which is normalized to have unit integral over phase space, and the expectation value of the quantity represented by A, for the state represented by F, is

$$(A,F) = \int A(q,p)F(q,p)dqdp\,,$$

where $\int dqdp$ means integration over all of phase space. We use the Heisenberg picture of dynamical transformations and transformations of reference

[21] E. Inönu and E. P. Wigner, Nuovo Cimento 9, 705 (1952).
[22] A. Loinger, Ann. Phys. (N.Y.) 20, 132 (1962).

frames. The same function F represents the state of the system at every time with respect to every reference frame, while the quantity represented by $A(q,p)$ at time zero with respect to a given frame is represented at time zero with respect to a transformed frame by the image $e^{[T]s}(A)(q,p)$ of $A(q,p)$ under the canonical transformation generated by T that represents the transformation of frames. [Here $T(q,p)$ is also, of course, a real function on phase space, the Lie bracket is the Poisson bracket

$$[A,T] = [A(q,p),T(q,p)]_{\mathrm{P.b.}} = \sum_{n=1}^{N}\sum_{i=1}^{3}\left(\frac{\partial A}{\partial q_i^n}\frac{\partial T}{\partial p_i^n} - \frac{\partial A}{\partial p_i^n}\frac{\partial T}{\partial q_i^n}\right)$$

and $e^{[T]s}(A)(q,p)$ is the solution of the differential equation

$$(\partial/\partial s)e^{[T]s}(A)(q,p) = [e^{[T]s}(A)(q,p),T(q,p)]_{\mathrm{P.b.}}$$

satisfying the boundary condition $e^{[T]s}(A)(q,p)|_{s=0} = A(q,p)$.] We define

$$Q_i^n(q,p) = q_i^n\,, \quad P_i^n(q,p) = p_i^n$$

for $n = 1,2,\cdots N$ and $i = 1,2,3$ as the functions whose values at a point of phase space are the coordinates of that point. We then have that

$$e^{[T]s}(A)(q,p) = A(e^{[T]s}(Q_1^1)(q,p),\cdots e^{[T]s}(Q_3^N)(q,p),\; e^{[T]s}(P_1^1)(q,p),\cdots e^{[T]s}(P_3^N)(q,p))\,,$$

which we write as

$$e^{[T]s}(A)(q,p) = A(e^{[T]s}(Q)(q,p),e^{[T]s}(P)(q,p))\,.$$

In other words, a canonical transformation of the functions on phase space can be represented by the canonical transformation of the phase space coordinates. The fact that a canonical transformation of phase space is measure preserving means that

$$\int e^{[T]s}(A)(q,p)dqdp = \int A(q,p)dpdp\,.$$

Let Q_i^n represent the ith component of the position of the nth particle at time zero with respect to a given frame of reference. Then, if H represents the Hamiltonian of the system with respect to this frame, the functions $e^{[H]\tau}(Q_i^n)(q,p)$ represent the positions of the particles at time $t = \tau$. For the case in which the state of the system is a pure state with the probability distribution concentrated on the point $(q_0,\, p_0)$ of phase space,

$$F(q,p) = \delta^{3N}(q - q_0)\delta^{3N}(p - p_0)\,,$$

the positions of the particles at time t have the exact values

$$q_i^n(t) = \int e^{[H]t}(Q_i^n)(q,p)\delta^{3N}(q - q_0)\delta^{3N}(p - p_0)dqdp = e^{[H]t}(Q_i^n)(q_0,p_0)\,. \tag{3.1}$$

[These are exact values as well as expectation values because the kth powers of the positions at time t have expectation values for this state which are equal to the kth powers of the expectation values (3.1), as can easily be seen.] Now the function $e^{[H]t}(Q_i^n)$ obeys the equation of motion

$$\begin{aligned}(\partial/\partial t)e^{[H]t}(Q_i^n)(q,p) &= [e^{[H]t}(Q_i^n)(q,p),H(q,p)] \\ &= [e^{[H]t}(Q_i^n)(q,p),e^{[H]t}(H)(q,p)] \\ &= e^{[H]t}([Q_i^n(q,p),H(q,p)]) \\ &= e^{[H]t}(\partial H/\partial p_i^n)(q,p)\end{aligned} \tag{3.2}$$

from which we can deduce the equations of motion satisfied by the functions $q_i^n(t)$ which describe the trajectories of the particles in space. To do this we introduce the functions

$$p_i^n(t) = e^{[H]t}(P_i^n)(q_0,p_0) \tag{3.3}$$

which can be treated mathematically in complete analogy to the functions (3.1). Corresponding to Eq. (3.2) we find that

$$(\partial/\partial t)e^{[H]t}(P_i^n)(q,p) = -e^{[H]t}(\partial H/\partial q_i^n)(q,p) \tag{3.4}$$

and from Eqs. (3.2) and (3.4), using the definitions (3.1) and (3.3), we obtain the equations

$$(\partial/\partial t)q_i^n(t) = [\partial/\partial p_i^n(t)]H(q(t),p(t)) \tag{3.5}$$

$$(\partial/\partial t)p_i^n(t) = -[\partial/\partial q_i^n(t)]H(q(t),p(t))\,, \tag{3.6}$$

which are just Hamilton's equations for the trajectories of the particles. In the case of pure state density distributions our formalism is thus equivalent to ordinary Hamiltonian mechanics. It is capable of describing, through the appropriate function $H(q,p)$, any interaction that gives particle trajectories describable by Hamilton's equations. This description will be relativistically symmetric (as described in the preceding section) if we can find nine other functions **P**, **J**, and **K** which together with H satisfy the Poisson bracket relations of generators of the relativistic transformation group.

We will see, however, that there are further questions that naturally arise when we transform our description to another reference frame and which suggest that additional assumptions are necessary in a theory describing relativistically invariant particle motion. With respect to a transformed frame of reference the positions of the particles are represented at time $t' = r$ by the functions

$$e^{[\exp([T]s)(H)]r}(e^{[T]s}(Q_i^n))(q,p) = e^{[H']r}(Q_i'^n)(q,p) \quad (3.7)$$

$[A'(q,p) = e^{[T]s}(A)(q,p)$ for any function $A(q,p)$; as an example of functions (3.7) see the description of noninteracting particles in Appendix A]. For example, if $T = J_3$, the second frame is rotated by an angle s about the z axis with respect to the initial frame. For the same pure state represented by the probability distribution concentrated at the point (q_0,p_0) as before, we obtain the trajectories

$$q_i'^n(t') = e^{[H']t'}(Q_i'^n)(q_0,p_0) \quad (3.8)$$

for the particles with respect to the second frame of reference. Now Eqs. (3.1) and Eqs. (3.8) give the values of the position coordinates of the particles as a function of time with respect to the two frames of reference. The question now is whether these quantities are related as we expect particle position coordinates to be related under a transformation of reference frames.

If $T = P_1$, corresponding to a displacement of the reference frame by a distance s in the x direction, we expect that

$$q_j'^n(t) = q_j^n(t) + s\delta_{j1}\,. \quad (3.9)$$

Since $[H,P_1] = 0$ implies $H' = e^{[P_1]s}(H) = H$, we have that

$$\begin{aligned} q_j'^n(t) &= e^{[H]t}(e^{[P_1]s}(Q_j^n))(q_0,p_0) \\ &= e^{[H]t}(Q_j^n)(q_0,p_0) \\ &\quad + se^{[H]t}([Q_j^n,P_1])(q_0,p_0) + \cdots, \end{aligned}$$

from which we see that the necessary and sufficient condition for Eq. (3.9) to be satisfied is that

$$[Q_j^n,P_1] = \delta_{j1}\,.$$

Similarly, if $T = J_3$, corresponding to a rotation of the frame by the angle s about the z axis, the necessary and sufficient condition, assuming that $[J_j, H] = 0$, in order that

$$\begin{aligned} q_1'^n(t) &= q_1^n(t)\cos s - q_2^n(t)\sin s\,, \\ q_2'^n(t) &= q_1^n(t)\sin s + q_2^n(t)\cos s\,, \end{aligned} \quad (3.10)$$

is that

$$\begin{aligned} [J_3,Q_1^n] &= Q_2^n \\ -[J_3,Q_2^n] &= Q_1^n\,. \end{aligned}$$

Thus, for either Lorentz or Galilean symmetric theories, in which the generating functions satisfy Eqs. (2.10), it is natural to require that the equations

$$[Q_j^n,P_k] = \delta_{jk} \quad (3.11)$$

$$[J_j,Q_k^n] = \epsilon_{jkl}Q_l^n \quad (3.12)$$

be satisfied for all $n = 1,2,\cdots N$ and $j,k = 1,2,3$ by the functions representing the position coordinates of the particles.

We can investigate the change of positions of the particles under transformations to uniformly moving reference frames in an identical manner, but for this purpose we must, of course, distinguish between Lorentz and Galilean transformations. We consider the Galilean case first and assume that the functions H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ satisfy Eqs. (2.10) and (2.12). If $T = K_1$, that is, if the second frame is moving with a uniform velocity s in the x direction with respect to the initial frame, the Galilean transformation of the particle positions is given by the equations

$$q_j'^n(t) = q_j^n(t) - \delta_{j1}st \quad (3.13)$$

which, since $(q_0,\ p_0)$ is an arbitrary point of phase space, are equivalent to the equations

$$e^{[K_1]s}(e^{[H]t}(Q_j^n)) = e^{[H]t}(Q_j^n) - \delta_{j1}st$$

in terms of functions on phase space. Now at $s = 0$ the above equations are identically true, so they are equivalent to the equations that result from taking the derivative with respect to s, namely,

$$[e^{[H]t}(Q_j^n),K_1] = -\delta_{j1}t$$

which in turn are equivalent to

$$\begin{aligned} -\delta_{j1}t &= [Q_j^n,e^{[H](-t)}(K_1)] \\ &= [Q_j^n,K_1 - tP_1] \\ &= [Q_j^n,K_1] - \delta_{j1}t \end{aligned}$$

if we assume Eq. (3.11). Therefore, in a theory which exhibits symmetry under the Galilei group in the ten generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ satisfying Eqs. (2.10) and (2.12), the necessary and sufficient condition for the particle positions to have the familiar transformation properties of the type (3.9), (3.10), and (3.13) is that the functions Q_j^n representing the particle positions satisfy Eqs. (3.11) and (3.12) and

$$[Q_j^n,K_k] = 0 \quad (3.14)$$

for all $n = 1,2,\cdots N$ and $j,k = 1,2,3$.

Now, for the case of Lorentz transformations, we assume that the functions H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ satisfy Eqs. (2.10) and (2.11) and we assume that the transformations of the particle positions under space translation and rotation are of the type (3.9) and (3.10) so that Eqs. (3.11) and (3.12) are also required to be valid. We then ask what further assumptions are needed to insure that the changes of particle positions under a transformation to a frame of reference moving uniformly with a velocity v in the x

direction with respect to the given frame are given by

$$q_1'^n(t') = [q_1^n(t_n) - vt_n]/(1-v^2)^{1/2}$$
$$q_j'^n(t') = q_j^n(t_n) \text{ for } j \neq 1$$
$$t' = [t_n - vq_1^n(t_n)]/(1-v^2)^{1/2}$$

or, equivalently, by

$$q_1'^n(t') = q_1^n(t_n)\cosh s - t_n \sinh s$$
$$q_j'^n(t') = q_j^n(t_n) \text{ for } j \neq 1$$
$$t' = t_n \cosh s - q_1^n(t_n)\sinh s \qquad (3.15)$$

where $s = \tanh^{-1} v$. We write these equations in the latter form because, considered as transformations of the particle positions, they form a one-parameter group in the parameter s. In fact, the transformations of particle positions under space translation and rotation and transformation to a moving frame—of the form (3.9), (3.10), and (3.15), together with transformations under translation in time of the form

$$q_j'^n(t) = q_j^n(t+s)$$

form a ten-parameter Lie group which is a realization of the Lorentz group.[8] In order for this group of transformations to be identical to the group of transformations of particle positions resulting from the group of automorphisms of phase space functions generated by H, **P**, **J**, and **K**, it is necessary and sufficient that it be identical to first order for each one-parameter subgroup.[13] This is the reason that the desired transformation properties of the particle positions under space translations and rotations and Galilean transformations could be obtained by specifying the Lie brackets (3.11), (3.12), and (3.14) of Q_j^n with **P**, **J**, and **K**. In these cases it was rather simple to see how the quantities behave under the whole one-parameter group, but for the more complicated case of Lorentz transformations we rely on the fact that we only need to work to first order in the group parameter. To first order in s then we have that

$$\begin{aligned} q_j'^n(r) &= e^{[\exp([K_1]s)(H)]r}(e^{[K_1]s}(Q_j^n))(q_0,p_0) \\ &= e^{[H]r\cosh s}(e^{[P_1](-r\sinh s)}(e^{[K_1]s}(Q_j^n)))(q_0,p_0) \\ &= e^{[H]r}(e^{[P_1](-rs)}(Q_j^n + s[Q_j^n,K_1]))(q_0,p_0) \\ &= e^{[H]r}(Q_j^n + s[Q_j^n,K_1] - rs[Q_j^n,P_1])(q_0,p_0) \\ &= q_j^n(r) + se^{[H]r}([Q_j^n,K_1])(q_0,p_0) - \delta_{j1}rs\,, \end{aligned} \qquad (3.16)$$

while from Eqs. (3.15) to first order in s we have

$$q_j'^n(t') = q_j^n(t_n) - \delta_{j1}t_n s$$
$$t' = t_n - q_1^n(t_n)s\,,$$

from which we obtain

$$q_j'^n(t') = q_j^n(t') + sq_1^n(t')(d/dt')q_j^n(t') - \delta_{j1}t's \qquad (3.17)$$

to first order in s. If Eqs. (3.16) and (3.17) are to be identical expressions for the functions $q_j'^n$ we must have that

$$\begin{aligned} e^{[H]r}([Q_j^n,K_1])(q_0,p_0) &= q_1^n(r)(d/dr)q_j^n(r) \\ &= e^{[H]r}(Q_1^n[Q_j^n,H])(q_0,p_0) \end{aligned}$$

or equivalently, since (q_0, p_0) is an arbitrary point of phase space, we have deduced that the equation

$$[Q_j^n,K_1] = Q_1^n[Q_j^n,H]$$

is a necessary and sufficient condition in order that the particle positions transform according to Eqs. (3.15) under Lorentz transformations. Thus, in a theory which exhibits symmetry under the Lorentz group in the ten generators H, **P**, **J**, and **K** satisfying the Lie bracket relations (2.10) and (2.11), the necessary and sufficient condition for the particle positions to have the familiar transformation properties of the type (3.9), (3.10), and (3.15) is that the functions Q_j^n representing the particle positions satisfy Eqs. (3.11) and (3.12) and

$$[Q_j^n,K_k] = Q_k^n[Q_j^n,H] \qquad (3.18)$$

for all $n = 1,2,\cdots N$ and $j,k = 1,2,3$. (As we have previously mentioned, one of the authors has derived essentially these same equations by considering only the geometrodynamical properties of the world lines of the particles.[8] This alternative method thus avoids making any assumption about the validity of our general dynamical formalism.) We note that unlike the other equations characterizing the transformation properties of particle positions, Eq. (3.18) involves the Lie bracket with two different generators, corresponding to the relation between time translation and Lorentz transformation, and is not linear in the functions representing particle coordinates. It requires not only the definition of the Lie bracket but also the definition of "ordinary" multiplication of the quantities in R.

In summary then, a relativistic classical mechanical description of the motion of a fixed number N of particles seems to require a theory involving two different kinds of assumptions. First, the theory is required to be symmetric under the group of transformations of reference frames. This requirement is satisfied in our formalism in the assumption that the theory contains the ten generators H, **P**, **J**, and **K** satisfying the Lie bracket equations (2.10) and (2.11) or (2.12) of the Lorentz or Galilei group, respectively.

Secondly, it is required that the particle positions transform in the familiar manner. This "manifest invariance" requirement takes the form of the additional assumption that the generators and the functions representing the particle positions satisfy Eqs. (3.11), (3.12), and (3.18) or (3.14) for the Lorentz or Galilei case, respectively. Whereas the first condition alone allows theories which include a large class of interactions between the particles, the combination of the two conditions may restrict the Lorentz symmetric theory so severely as to rule out any interaction, as we will see for the case of two particles in the next section.

Our argument in establishing the conditions (3.11), (3.12), and (3.18) or (3.14) characteristic of particle positions in a Lorentz or Galilean symmetric theory was based on classical mechanics. What about quantum mechanics? The situation in quantum mechanics is much less clear because there a particle does not have a definite trajectory; that is, an exact value for its position at each time. We would like to discuss two of the many possible points of view that one can take. One can argue that the particle position conditions (3.11), (3.12), and (3.18) or (3.14) are equations between dynamical variables which, being independent of the specification of the state of the system, should be maintained whether the states are defined as in classical mechanics or as in quantum mechanics. These equations are part of the structure of the algebra of dynamical variables, just as are Eqs. (2.10) and (2.11) or (2.12) which specify the structure of the Lie algebra of generators. This structure must be the same in both classical and quantum mechanics; the only change to be made in transition is to replace Poisson brackets with commutators.[23] In particular, if the equations involving the Lie brackets of the particle position with the momentum and angular momentum are to be maintained, then the equations involving the Lie brackets of the particle position with the generators of transformations to uniformly moving frames must also be maintained. [Note, however, that the fact that Eq. (3.18) involves a product of functions distinguishes this equation in an essential manner in this regard from the other equations characteristic of particle positions. To transcribe Eq. (3.18) into a quantum-mechanical equation one needs to assume not only that Poisson brackets are to be replaced with commutator brackets but also that products of functions are to be replaced by symmetrized products of operators.[18]] The above point of view is, in fact, the one that is customarily followed in constructing the generators for a quantum mechanical description of a fixed number of noninteracting particles (see Appendix A).

On the other hand, one can argue that in order to be valid for quantum mechanics these particle position conditions must be established, just as they were for classical mechanics, from the transformation properties of space–time events. If we can not use exact trajectories in our argument then we must use the next best thing; for example, the expectation value of the particle position as a function of time. Thus if we define

$$\begin{aligned} q_j^n(t) &= \langle e^{[H]t}(Q_j^n)\rangle \\ &= \langle\psi|e^{iHt}Q_j^n e^{-iHt}|\psi\rangle \end{aligned}$$

as the expectation value of $e^{[H]t}(Q_j^n)$ for a quantum-mechanical pure state represented by the vector ψ, we can require that $q_j^n(t)$ transform in the familiar manner under changes of reference frame. For example, with respect to a frame translated a distance s in the x direction from the initial frame, the average particle positions are

$$\begin{aligned} q_j'^n(t) &= \langle e^{[H]t}(e^{[P_1]s}(Q_j^n))\rangle \\ &= \langle e^{[H]t}(Q_j^n)\rangle + s\langle e^{[H]t}([Q_j^n,P_1])\rangle + \cdots, \end{aligned}$$

which we require to agree with those given by Eq. (3.9). As a consequence, Eqs. (3.11) are established for quantum mechanics just as they were for classical mechanics. Similarly, Eqs. (3.12) can be established for quantum mechanics by requiring that the changes of the average values of the particle positions under rotation of the reference frame be of the type (3.10). For the Galilean case, Eqs. (3.14) also follow from the requirement that the average particle positions have the familiar transformation property—in this case that of the type (3.13). In each of these cases the desired operator equations follow from equations satisfied by the expectation values for every pure state. The success of this procedure is based on the fact that the latter equations are linear in the expectation values—a fact that fails to hold for Lorentz transformations.

The desired change of average particle positions under Lorentz transformation to a frame moving uniformly in the x direction with velocity $v = \tanh s$ is given by Eq. (3.17) to first order in the group parameter s, while, in complete analogy to Eq. (3.16), we have that

$$q_j'^n(t') = q_j^n(t') + s\langle e^{[H]t'}([Q_j^n,K_1])\rangle - \delta_{j1}t's\,.$$

[23] The structure of the Lie algebra as functions of the canonical variables will not, however, be the same for quantum mechanics as for classical mechanics. See reference 18.

The condition that the latter expression be identical to Eq. (3.17) for every pure state is that

$$\langle e^{[H]t'}([Q_j^n,K_1])\rangle = \langle e^{[H]t'}(Q_1^n)\rangle\langle e^{[H]t'}([Q_j^n,H])\rangle \quad (3.19)$$

for every pure state, or equivalently, absorbing the time dependence into the state, that

$$\langle [Q_j^n,K_1]\rangle = \langle Q_1^n\rangle\langle [Q_j^n,H]\rangle \quad (3.20)$$

for every pure state. This nonlinear equation does not serve to establish an operator equation of the type (3.18). In fact, for given operators Q_j^n and H it is not possible, in general, to define any operator $[Q_j^n, K_1]$ by the condition that the above equation be satisfied by the expectation value for every pure state. (It is also instructive to consider the situation in the usual quantum-mechanical description of noninteracting particles. In this case, one finds that the average particle positions transform according to the usual formula under Lorentz transformations only when the expectation values are taken for a pure state that is an "eigenstate" of the operators $[Q_j^n, H]$. See Appendix A.)

Thus, we find that we have a natural method for establishing the equations for the Lie brackets of the particle positions Q_j^n with the total momentum, total angular momentum, and generators of Galilei transformations, but that this method breaks down for generators of Lorentz transformations. This failure to generalize Eq. (3.18), which is due to the fact that, in contrast to the other equations characteristic of particle positions, it involves a product of functions, could be viewed as a reason to proceed with the construction of quantum mechanical theories containing Eqs. (3.11) and (3.12) and in the case of Galilean symmetry Eq. (3.14), but not including Eq. (3.18) in the case of Lorentz symmetry.[24] However, we would tend to view such an attitude as having only the temporary validity of expediency. It is surely not completely or finally satisfactory to rest on our inability to treat the question of Lorentz transformation of a quantum-mechanical particle position as an excuse for ignoring the question. We would hope to be able to find a way to make a more decisive statement.

IV. CONSEQUENCES OF THE COMBINED INVARIANCE ASSUMPTIONS

We now investigate the implications for a classical mechanical theory of particle motion of the combined assumptions of Eqs. (2.10) and (2.11) or (2.12) which exhibit the symmetry of the theory under the Lorentz or Galilei group, and Eqs. (3.11), (3.12), and (3.18) or (3.14) which assure that the space–time events comprising the world line of a particle transform in the usual Einstein–Lorentz or Galilean manner. For this purpose we limit ourselves to the consideration of just two particles. All of the quantities entering our theory are functions of the twelve real variables $q_j^1, p_j^1, q_j^2, p_j^2, j=1,2,3$. The result, is, roughly, that the addition of Eqs. (3.11), (3.12), and (3.14) to Eqs. (2.10) and (2.12) imposes a rather harmless condition for a Galilean invariant interaction between the particles, while the combination of Eqs. (3.11), (3.12), and (3.18) with Eqs. (2.10) and (2.11) for a Lorentz invariant theory essentially rules out any interaction. One can not construct any functions H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ satisfying these latter equations except those functions that give constant velocities for both of the particles, or in other words give

$$[[Q_j^n,H],H] = 0 \quad (4.1)$$

for $n = 1,2; j = 1,2,3$.

First we will show that Eqs. (3.11) are equivalent to the assumption that the generators P_j have the standard "free-particle" form

$$P_j = P_j^1 + P_j^2 \quad (4.2)$$

for $j = 1,2,3$. Clearly, the generators given by (4.2) satisfy Eqs. (3.11). We want to show that any generators P_j that satisfy Eqs. (3.11) can be put in the form (4.2) by a suitable canonical transformation. If

$$P_j = P_j^1 + P_j^2 + F_j$$

it follows from Eqs. (3.11) that

$$[F_j,Q_k^n] = 0$$

for $n = 1,2; j,k, = 1,2,3$, so that F_j is a function only of q_k^1 and q_k^2. Let us make the change of variables

$$r_j = q_j^1 + q_j^2$$
$$q_j = q_j^1 - q_j^2 \quad (4.3)$$

in terms of which, using Eqs. (2.10), we find that

$$0 = [P_j,P_k] = 2(\partial F_j/\partial r_k - \partial F_k/\partial r_j) .$$

By using these new variables and the above equation it is easy to show[8] that we produce a canonical transformation if, while keeping Q_j^1 and Q_j^2 fixed, we transform P_j^1 and P_j^2 by

$$P_j^1 \to P_j^1 - \tfrac{1}{2} F_j - \tfrac{1}{2} W_j$$
$$P_j^2 \to P_j^2 - \tfrac{1}{2} F_j + \tfrac{1}{2} W_j$$

[24] It seems to us that this attitude is evident in the work of L. L. Foldy,[6] although Foldy does require that the particle positions have the familiar Galilei transformation properties in the limit as the velocity of light becomes infinite.

where

$$W_j = \tfrac{1}{3} \sum_k \int dr_k \frac{\partial F_k}{\partial q_j}.$$

The generator P_j undergoes the transformation

$$P_j = P_j^1 + P_j^2 + F_j \rightarrow P_j^1 + P_j^2$$

and the other generators H, $\mathbf{J}$, and $\mathbf{K}$ are also transformed to new functions of q_j^n and p_j^n. If we take these new functions to represent the generators of transformations of reference frames we obtain a new representation for our theory which is identical in content to the given representation. But now the generators P_j have the form (4.2) which from now on we, therefore, assume.

Let us explain exactly why we say that we can make the above canonical transformation on the generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ without changing the physical content of the theory. Clearly, the new generators satisfy the same Poisson bracket equations (2.10) and (2.11) or (2.12) as the given generators. Since the functions Q_j^n are unchanged, the Poisson brackets of these functions with the generators, and thus the transformation properties of the particle positions, are also preserved. Consider, for the moment, any given canonical transformation and let a primed function represent the image of that function under the given canonical transformation. Then, for example, $Q_j'^n$ and H' give the same set of particle trajectories as Q_j^n and H. For

$$e^{[H']t}(Q_j'^n) = (e^{[H]t}(Q_j^n))'$$

and any (pure state) density function is the image F' under the given canonical transformation of a (pure state) density function F. Thus, the motion of the system as calculated from $Q_j'^n$ and H' with respect to the initial conditions represented by F' is described by the particle trajectory functions

$$\int e^{[H']t}(Q_j'^n)F'dqdp = \int e^{[H]t}(Q_j^n)Fdqdp\,,$$

which are the same as the particle trajectory functions describing the motion of the system as calculated from Q_j^n and H with the initial conditions represented by F. The canonical transformation leaves invariant the set of all allowed particle motions corresponding to the set of all allowed density functions. These considerations can easily be generalized to include the transformations generated by the other nine functions $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ which give the particle trajectories with respect to transformed reference frames. Since we are interested only in the motion of the particles as described by the time dependence of their positions in space, we consider the canonically transformed description to have the same physical content as the original description. In the particular case that $Q_j'^n$ equals Q_j^n, as is true above, we consider the new generators H', $\mathbf{P}'$, $\mathbf{J}'$, and $\mathbf{K}'$, as functions of q_j^n and p_j^n, to describe the physically same theory as the original generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$, while we maintain our identification of Q_j^n as representing the positions of the particles. If $Q_j'^n$ did not equal Q_j^n we would have to let $Q_j'^n$ and not Q_j^n represent the positions of the particles in order to obtain an equivalent description by the canonically transformed function, and we would no longer be interested in the form of the canonically transformed generators as functions of q_j^n and the conjugate variables p_j^n. [An alternative procedure would be to also identify the functions P_j^n with the particle momenta, in which case the freedom to do canonical transformations that change these functions would be lost. This alternative is treated in Appendix D. If one requires that the P_j^n be unchanged under space translations, as is familiar for particle momenta, it follows that the generators P_j have the "free-particle" form (4.2); the canonical transformations are not needed. By means of canonical transformations we can secure the same result from a weaker assumption.]

Next we show that Eqs. (3.12) are equivalent to the assumption that the generators J_j have the standard "free-particle" form

$$J_j = \epsilon_{jkl}Q_k^1P_l^1 + \epsilon_{jkl}Q_k^2P_l^2 \tag{4.4}$$

for $j = 1,2,3$. Again it is clear that the generators J_j given by (4.4) satisfy Eqs. (3.12) so our task is to show that any generators J_j satisfying Eqs. (3.12) can be put in the form (4.4) by a suitable canonical transformation. Again we want to keep the functions Q_j^n fixed to preserve our identification of these functions as representing the particle positions. Now we also want to keep fixed the generators P_j which we assume to have the form (4.2). Suppose then that

$$J_j = \epsilon_{jkl}Q_k^1P_l^1 + \epsilon_{jkl}Q_k^2P_l^2 + F_j\,.$$

From Eqs. (3.12) it follows that

$$[F_j,Q_k^n] = 0\,,$$

so that F_j is a function only of $\mathbf{q}^1$ and $\mathbf{q}^2$. From Eq. (2.10) that

$$[J_j,P_k] = \epsilon_{jkl}P_l$$

it follows that

$$[F_j,P_k] = 0\,,$$

so that if we use the explicit form (4.2) of P_j and the change of variables (4.3) we deduce that F_j is a function only of the three variables q_j. Finally from Eq. (2.10) that

$$[J_j,J_k] = \epsilon_{jkl}J_l\,,$$

it follows that

$$\epsilon_{kmn}q_m(\partial F_j/\partial q_n) - \epsilon_{jmn}q_m(\partial F_k/\partial q_n) = \epsilon_{jkl}F_l$$

which, letting ∇ be the gradient operator with respect to the q coordinates, we can write in vector notation as

$$(\mathbf{q} \times \nabla) \times \mathbf{F} = -\mathbf{F}$$

or, after some elementary manipulation with vector identities, as

$$\mathbf{q}(\nabla\cdot\mathbf{F}) = \nabla(\mathbf{q}\cdot\mathbf{F})\,.$$

Now it can be shown[25] that if $\mathbf{F}$ satisfies the above equation, there exists a function W of the variables $\mathbf{q}$ such that

$$\mathbf{F} = \mathbf{q} \times \nabla W\,.$$

If we use this result it is easy to show[8] that we produce a canonical transformation by keeping Q_j^1 and Q_j^2 fixed and transforming P_j^1 and P_j^2 by

$$P_j^1 \to P_j^1 - \nabla_j W$$
$$P_j^2 \to P_j^2 + \nabla_j W\,.$$

Under this transformation the form (4.2) of P_j is preserved and

$$\mathbf{J} = \mathbf{Q}^1 \times \mathbf{P}^1 + \mathbf{Q}^2 \times \mathbf{P}^2 + \mathbf{F}$$

is transformed to

$$\begin{aligned}\mathbf{Q}^1 &\times (\mathbf{P}^1 - \nabla W) + \mathbf{Q}^2 \times (\mathbf{P}^2 + \nabla W) + \mathbf{F}\\ &= \mathbf{Q}^1 \times \mathbf{P}^1 + \mathbf{Q}^2 \times \mathbf{P}^2 - (\mathbf{Q}^1 - \mathbf{Q}^2) \times \nabla W + \mathbf{F}\\ &= \mathbf{Q}^1 \times \mathbf{P}^1 + \mathbf{Q}^2 \times \mathbf{P}^2\,,\end{aligned}$$

so that the generators J_j have been put in the form (4.4). [If one also identifies the P_j^n with the particle momenta and requires that they transform as vectors under space rotations, one can deduce the "free-particle" form (4.4) for the generators J_j without making any canonical transformations. See Appendix D.] From now on we, therefore, assume that both P_j and J_j have the standard "free-particle" forms (4.2) and (4.4), respectively. In showing that this assumption is equivalent to Eqs. (3.11) and (3.12) we have used only Eqs. (2.10), so what we have done so far is valid for both the Lorentz and Galilei cases.

To investigate the implications of the remaining condition on the particle positions, namely, that involving the Poisson bracket of Q_j^n with K_k, we turn first to the case of Galilean invariance. Our problem is to find the class of functions H (or, in other words, the class of interactions) that allow Eqs. (2.10) and (2.12) to be satisfied by the generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ when $\mathbf{P}$ and $\mathbf{J}$ are assumed to have the "free-particle" forms (4.2) and (4.4) and $\mathbf{K}$ is assumed to satisfy Eq. (3.14). Let us write

$$\begin{aligned}H &= (1/2m_1)(\mathbf{P}^1)^2 + (1/2m_2)(\mathbf{P}^2)^2 + V\\ &= H^0 + V\end{aligned}$$

as the sum of the "free-particle" Hamiltonian H^0 and the potential V and similarly

$$\begin{aligned}K_j &= m_1Q_j^1 + m_2Q_j^2 + W_j\\ &= K_j^0 + W_j\end{aligned}$$

as the sum of the "free-particle" Galilean generators $\mathbf{K}^0$ and an additional vector function $\mathbf{W}$. If we substitute these expressions into Eqs. (2.10) and (2.12), and use the fact that the "free-particle" generators H^0, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}^0$ satisfy these equations, we find that V must have a vanishing Poisson bracket with each of the generators $\mathbf{P}$ and $\mathbf{J}$, which means that V must be a scalar function that is independent of the variables r_j of (4.3). In other words, V must be a function of the scalars that can be formed from the three vectors $\mathbf{p}^1$, $\mathbf{p}^2$, and $\mathbf{q}$. We also find that V is further limited by having to satisfy the condition that

$$[V,K_j^0] = [W_j,H] \tag{4.5}$$

and we find that W_j must have the form

$$W_j = \tfrac{1}{2}(M - m_1 - m_2)r_j + R_j\,, \tag{4.6}$$

where M is the neutral element of Eq. (2.12) and $\mathbf{R}$ represents functions of $\mathbf{p}^1$, $\mathbf{p}^2$, and $\mathbf{q}$ that transform as a vector under space rotation, that is, satisfy the equations

$$[J_j,R_k] = \epsilon_{jkl}R_l\,. \tag{4.7}$$

These functions must also satisfy the equations

$$[R_j,P_k] = 0\,, \tag{4.8}$$

and are further limited in that the equation

$$[K_j,K_k] = 0 \tag{4.9}$$

must also be satisfied. Under the conditions we have listed, the generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ obey Eqs. (2.10) and (2.12).

The added requirement that is imposed by the assumption that the generators K_j satisfy Eq. (3.14)

[25] J. B. Keller, Commun. Pure Appl. Math. 14, 77 (1961), Lemma 2.

is simply that K_j be independent of $\mathbf{p}^1$ and $\mathbf{p}^2$. This means that R_j must have the form

$$R_j = q_j R ,$$

where R is a function of the scalar $(\mathbf{q})^2$. Here we have used Eqs. (4.7) and (4.8) which are now automatically satisfied as is Eq. (4.9). In summary, then, any potential V is allowed which is a scalar function of the variables $\mathbf{p}^1$, $\mathbf{p}^2$, and $\mathbf{q}$ and satisfies Eq. (4.5), where

$$W_j = \tfrac{1}{2}(M - m_1 - m_2)r_j + q_j R$$

with R a function of the scalar $(\mathbf{q})^2$. These are, in fact, the only restrictions on the interaction that we can derive from our assumptions of Galilean symmetry and Galilean transformation of the particle positions.

Suppose that $W_j = 0$. Then from Eq. (4.5) we see that V has a vanishing Poisson bracket with the center-of-mass coordinates

$$(1/M)K_j^0 = (1/M)(m_1 Q_j^1 + m_2 Q_j^2) , \quad (4.10)$$

which means that V is independent of $(1/m_1)p_j^1 + (1/m_2)p_j^2$. For the equal-mass case $m_1 = m_2$, we have that V is independent of the total momentum $p_j^1 + p_j^2$ and depends on the p_j^n variables only through the relative variable $p_j^1 - p_j^2$. This result has been used by Eisenbud and Wigner[26] and by Okubo and Marshak[27] in deriving the "most general" form of the two-nucleon interaction.

However W_j need not be zero; there is a large class of generators involving nonzero W_j which satisfy all of our conditions. Furthermore, we do not feel justified in setting W_j equal to zero by a suitable canonical transformation.[28] The reason for this is that K_j is a function only of the q_j^n and not of the p_j^n. To change the form of K_j we would need to do a canonical transformation that changes the Q_j^n, and we have chosen to let the functions Q_j^n represent the positions of the particles. If the generators are to be rerepresented by the images of H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ under a canonical transformation, then the particle positions must be rerepresented by the image of Q_j^n, and the image of K_j will have the same form as a function of the images of the Q_j^n as K_j as a function of Q_j^n.

[26] L. Eisenbud and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. 27, 281 (1941).

[27] S. Okubo and R. E. Marshak, Ann. Phys. (N.Y.) 4, 166 (1958).

[28] Such a canonical transformation is used by L. L. Foldy,[6] to reduce K_j to the standard form.

We can find a justification for the requirement that V have a vanishing Poisson bracket with the center-of-mass coordinates (4.10) in an apparently reasonable additional assumption. The "time derivatives" of these coordinates are

$$[(1/M)(m_1 Q_j^1 + m_2 Q_j^2),H] = (1/M)P_j + [(1/M)(m_1 Q_j^1 + m_2 Q_j^2),V] ,$$

so the condition that the center-of-mass coordinate (4.10) moves with uniform velocity $(1/M)\mathbf{P}$ is that it have a vanishing Poisson bracket with V. Note that this does not require that W_j vanish but only [by Eq. (4.5)] that W_j have a vanishing Poisson bracket with H. (If one identifies the P_j^n with the particle momenta and requires that they transform under Galilei transformations to uniformly moving reference frames in the manner that is familiar for momenta, one can immediately deduce that $W_j = 0$ or that K_j has the standard "free-particle" form. The criterion of Eisenbud and Wigner is then established. See Appendix D.)

Finally, we turn to our main task which is to find the implications of Eq. (3.18) for a classical mechanical theory of two particles in which the symmetry under the Lorentz group is exhibited in the ten generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ satisfying Eqs. (2.10) and (2.11). We assume that $\mathbf{P}$ and $\mathbf{J}$ have already been put in the forms (4.2) and (4.4) by the assumption of Eqs. (3.11) and (3.12). In contrast to the Galilean case where Eq. (3.14) imposed rather mild restrictions on the interaction, we find that in the Lorentz case Eq. (3.18) has the devastating consequence of eliminating any nontrivial interaction. This is also to be contrasted with the rather mild restriction implied by Eqs. (3.11) and (3.12) in the absence of Eq. (3.18). The large class of Lorentz symmetric classical and quantum-mechanical interactions constructed by Bakamjian and Thomas[5] and by Foldy[6] allow the identification of $\mathbf{P}$ and $\mathbf{J}$ in the standard "free-particle" forms (4.2) and (4.4).

We begin then with Eq. (3.18) which we write as

$$\partial K_k/\partial p_j^n = Q_k^n(\partial H/\partial p_j^n) \qquad (4.11)$$

for $n = 1,2; j,k = 1,2,3$. If we take the derivative of this equation with respect to p_l^m and compare with the result obtained by taking the derivative with respect to p_j^n of the equation obtained from Eq. (4.11) by changing the indices n and j to m and l, we find that

$$(Q_k^n - Q_k^m)\partial^2 H/\partial p_j^n \partial p_l^m = 0$$

from which we deduce, for the case $m \neq n$, that[29]

$$\partial^2 H/\partial p_j^1 \partial p_l^2 = 0$$

for $j,l = 1,2,3$. This means that

$$H = H_1 + H_2 \tag{4.12}$$

where H_1 is a function only of $\mathbf{p}^1$, $\mathbf{q}^1$, and $\mathbf{q}^2$ and H_2 is a function only of $\mathbf{p}^2$, $\mathbf{q}^1$, and $\mathbf{q}^2$.

Equation (2.10) tells us that H must have a vanishing Poisson bracket with P_j, from which, by using the variables (4.3), we see that

$$\partial H_1/\partial r_j = -\partial H_2/\partial r_j \, .$$

Now, since the right-hand side is independent of p^1 and the left-hand side is independent of p^2, the above quantity must be a function only of $\mathbf{r}$ and $\mathbf{q}$. From the fact that

$$\epsilon_{ijk}(\partial^2 H_1/\partial r_j \partial r_k) = 0 \, ,$$

we then conclude that there exists a function G of $\mathbf{r}$ and $\mathbf{q}$ such that

$$\partial H_1/\partial r_j = -\partial H_2/\partial r_j = \partial G/\partial r_j \, .$$

Hence, we can redefine H_1 and H_2 by subtracting G from H_1 and adding G to H_2 so as to obtain a new division of H in the form (4.12) in which the new H_1 and H_2 both have vanishing Poisson brackets with P_j and are thus independent of r_j. We therefore assume that H_1 is a function only of $\mathbf{p}^1$ and $\mathbf{q}$ and that H_2 is a function only of $\mathbf{p}^2$ and $\mathbf{q}$.

From Eqs. (2.10) we also know that H must have a vanishing Poisson bracket with J_j or

$$[H_1,J_j] = -[H_2,J_j] \, .$$

As before, the right-hand side is independent of $\mathbf{p}^1$ and the left-hand side is independent of $\mathbf{p}^2$, and we can conclude that the above quantity is a function only of $\mathbf{q}$. Now, from the Jacobi identity and the Poisson bracket equations (2.10) for $\mathbf{J}$, we obtain

$$[J_i,[H_1,J_j]] - [J_j,[H_1,J_i]] = \epsilon_{ijk}[H_1,J_k]$$

which we can put into the vector notation

$$[H_1,\mathbf{J}] = -(\mathbf{q} \times \nabla) \times ([H_1,\mathbf{J}]) \, ,$$

where ∇ is the gradient operator with respect to the $\mathbf{q}$ variables. We have already made use of an equation of this form in our argument for the identification of $\mathbf{J}$ in the form (4.4) and we know that it implies that there exists a function W of the variables $\mathbf{q}$ such that[25]

$$[H_1,\mathbf{J}] = -[H_2,\mathbf{J}] = \mathbf{q} \times \nabla W \, ,$$

which is equivalent to

$$[H_1,J_j] = -[H_2,J_j] = [W,J_j] \, ,$$

as can easily be seen by substituting the form (4.4) for J_j and evaluating the Poisson bracket on the right in the above equation. This means that we can redefine H_1 and H_2 by subtracting W from H_1 and adding W to H_2 so as to obtain a new division of H in the form (4.12) in which the new H_1 and H_2 both have vanishing Poisson brackets with J_j and are therefore scalar functions. Without loss of generality we may assume that H_1 is a function only of $(\mathbf{p}^1)^2$, $\mathbf{p}^1 \cdot \mathbf{q}$, and $(\mathbf{q})^2$ and that H_2 is a function only of $(\mathbf{p}^2)^2$, $\mathbf{p}^2 \cdot \mathbf{q}$, and $(\mathbf{q})^2$.

If we substitute the form (4.12) for H back into Eq. (4.11) we find that

$$K_j = Q_j^1 H_1 + Q_j^2 H_2 + F_j \, ,$$

where F_j is a function only of q^1 and q^2. From the Eqs. (2.10) that

$$[K_j,P_k] = \delta_{jk} H \, ,$$

$$[J_i,K_j] = \epsilon_{ijk} K_k \, ,$$

combined with our knowledge of the form of H and $\mathbf{K}$, we conclude that

$$[F_j,P_k] = 0 \, ,$$

$$[J_i,F_j] = \epsilon_{ijk} F_k \, ,$$

which means that $\mathbf{F}$ must be independent of $\mathbf{r}$ and a vector function of $\mathbf{q}$ only, or more specifically, $F_j = q_j F$ where F is a function only of the scalar $(\mathbf{q})^2$. Then

$$K_j = Q_j^1(H_1 + F) + Q_j^2(H_2 - F) \, ,$$

so if we again redefine H_1 and H_2 by adding F to H_1 and subtracting F from H_2 so as to obtain a new division of H in the form (4.12), we can put $\mathbf{K}$ in the form

$$K_j = Q_j^1 H_1 + Q_j^2 H_2 \tag{4.13}$$

without destroying the properties we have assumed for H_1 and H_2. So far, we have shown that without any loss of generality we may assume that H has the form (4.12) and that $\mathbf{K}$ has the form (4.13) where H_1 is a function only of $(\mathbf{p}^1)^2$, $\mathbf{p}^1 \cdot \mathbf{q}$, and $(\mathbf{q})^2$ and H_2 is a function only of $(\mathbf{p}^2)^2$, $\mathbf{p}^2 \cdot \mathbf{q}$, and $(\mathbf{q})^2$.

Now if we take our derivatives of H_1 and H_2 with

[29] Our proof makes use of the requirement that all of the generators be sufficiently regular functions of the canonical variables. This rules out any generators that may contain singular terms, such as delta functions, that could represent contact interactions. To include the latter would require an extension of the usual classical mechanical formalism (see reference 8).

respect to these scalar variables and evaluate the Poisson bracket

$$[K_j,H] = H_1[Q_j^1,H_1] + H_2[Q_j^2,H_2] + (Q_j^1 - Q_j^2)[H_1,H_2]\,, \qquad (4.14)$$

we get a scalar function times $(Q_j^1 - Q_j^2)$ plus the terms

$$2H_1[\partial H_1/\partial(\mathbf{p}^1)^2]P_j^1 + 2H_2[\partial H_2/\partial(\mathbf{p}^2)^2]P_j^2\,.$$

But from Eq. (2.11) we have that

$$[K_j,H] = P_j = P_j^1 + P_j^2\,,$$

so that we must have

$$2H_1[\partial H_1/\partial(\mathbf{p}^1)^2] = 2H_2[\partial H_2/\partial(\mathbf{p}^2)^2] = 1\,,$$

which means that H_1 and H_2 must have the form

$$H_1 = [(\mathbf{p}^1)^2 + W_1]^{1/2}$$
$$H_2 = [(\mathbf{p}^2)^2 + W_2]^{1/2}$$

where W_1 is a function only of $\mathbf{p}^1\cdot\mathbf{q}$ and $(\mathbf{q})^2$ and W_2 is a function only of $\mathbf{p}^2\cdot\mathbf{q}$ and $(\mathbf{q})^2$. If we now evaluate the Poisson bracket (4.14), we get the desired terms $P_j^1 + P_j^2$ plus an additional term which equals $(Q_j^1 - Q_j^2)$ times the function

$$\tfrac{1}{2}\,\partial W_1/\partial(\mathbf{p}^1\cdot\mathbf{q}) + \tfrac{1}{2}\,\partial W_2/\partial(\mathbf{p}^2\cdot\mathbf{q}) + [H_1,H_2] \qquad (4.15)$$

and so we conclude that this function must vanish. It is a simple task to evaluate

$$-4H_1H_2[H_1,H_2] = (\mathbf{p}^1\cdot\mathbf{p}^2)2\left(\frac{\partial W_1}{\partial(\mathbf{p}^1\cdot\mathbf{q})} + \frac{\partial W_2}{\partial(\mathbf{p}^2\cdot\mathbf{q})}\right)$$
$$+ 2(\mathbf{q})^2\left(\frac{\partial W_1}{\partial(\mathbf{q})^2}\frac{\partial W_2}{\partial(\mathbf{p}^2\cdot\mathbf{q})} + \frac{\partial W_2}{\partial(\mathbf{q})^2}\frac{\partial W_1}{\partial(\mathbf{p}^1\cdot\mathbf{q})}\right)$$
$$+ (\mathbf{p}^1\cdot\mathbf{q})\left(4\frac{\partial W_2}{\partial(\mathbf{q})^2} + \frac{\partial W_1}{\partial(\mathbf{p}^1\cdot\mathbf{q})}\frac{\partial W_2}{\partial(\mathbf{p}^2\cdot\mathbf{q})}\right)$$
$$+ (\mathbf{p}^2\cdot\mathbf{q})\left(4\frac{\partial W_1}{\partial(\mathbf{q})^2} + \frac{\partial W_1}{\partial(\mathbf{p}^1\cdot\mathbf{q})}\frac{\partial W_2}{\partial(\mathbf{p}^2\cdot\mathbf{q})}\right) \qquad (4.16)$$

which must be equal to

$$2H_1H_2[\partial W_1/\partial(\mathbf{p}^1\cdot\mathbf{q}) + \partial W_2/\partial(\mathbf{p}^2\cdot\mathbf{q})] \qquad (4.17)$$

in order for (4.15) to be equal to zero. Now H_1, H_2, W_1, and W_2 are functions only of $(\mathbf{p}^1)^2$, $(\mathbf{p}^1\cdot\mathbf{q})$, $(\mathbf{p}^2)^2$, $(\mathbf{p}^2\cdot\mathbf{q})$, and $(\mathbf{q})^2$, and the variable $(\mathbf{p}^1\cdot\mathbf{p}^2)$ is independent of these. Therefore, since (4.16) and (4.17) must be equal, the coefficient of $(\mathbf{p}^1\cdot\mathbf{p}^2)$, which occurs only in the first term of (4.16), must vanish. Thus, we have that

$$\partial W_1/\partial(\mathbf{p}^1\cdot\mathbf{q}) = -\partial W_2/\partial(\mathbf{p}^2\cdot\mathbf{q})$$

and, since the right-hand side is independent of $(\mathbf{p}^1\cdot\mathbf{q})$ and the left-hand side is independent of $(\mathbf{p}^2\cdot\mathbf{q})$, this quantity must be a function only of $(\mathbf{q})^2$. Consequently, the functions W_1 and W_2 must have the form

$$W_1 = (\mathbf{p}^1\cdot\mathbf{q})F + G_1\,,$$
$$W_2 = -(\mathbf{p}^2\cdot\mathbf{q})F + G_2\,,$$

where G_1, G_2, and F are functions only of $(\mathbf{q})^2$. If we substitute these into (4.16) and (4.17) and equate the two resulting expressions, we obtain the equation

$$(\mathbf{p}^1\cdot\mathbf{q})[4G_2' - F^2 - 2(\mathbf{q})^2FF']$$
$$+ (\mathbf{p}^2\cdot\mathbf{q})[4G_1' - F^2 - 2(\mathbf{q})^2FF']$$
$$+ 2(\mathbf{q})^2F(G_2' - G_1') = 0$$

where prime denotes differentiation with respect to $(\mathbf{q})^2$. Since $(\mathbf{p}^1\cdot\mathbf{q})$ and $(\mathbf{p}^2\cdot\mathbf{q})$ are independent of $(\mathbf{q})^2$, each term in the above equation must separately vanish so we have that

$$G_1' = G_2' = \tfrac{1}{4}[F^2 + 2(\mathbf{q})^2FF'] = \tfrac{1}{4}[d/d(\mathbf{q})^2](\mathbf{q})^2F^2$$

which means that

$$G_1 = \tfrac{1}{4}(\mathbf{q})^2F^2 + m_1^2\,,$$
$$G_2 = \tfrac{1}{4}(\mathbf{q})^2F^2 + m_2^2\,,$$

where m_1^2 and m_2^2 are constants. Substituting these into the expressions for W_1 and W_2 and these in turn into the expressions for H_1 and H_2, we find that we have

$$H_1 = [(\mathbf{p}^1)^2 + (\mathbf{p}^1\cdot\mathbf{q})F + \tfrac{1}{4}(\mathbf{q})^2F^2 + m_1^2]^{1/2}$$
$$= [(\mathbf{p}^1 + \tfrac{1}{2}F\mathbf{q})^2 + m_1^2]^{1/2}$$
$$H_2 = [(\mathbf{p}^2)^2 - (\mathbf{p}^2\cdot\mathbf{q})F + \tfrac{1}{4}(\mathbf{q})^2F^2 + m_2^2]^{1/2}$$
$$= [(\mathbf{p}^2 - \tfrac{1}{2}F\mathbf{q})^2 + m_2^2]^{1/2} \qquad (4.18)$$

with F a function only of $(\mathbf{q})^2$.

We can now see that the Hamiltonian H is not capable of describing any interaction. For if we keep Q_j^1 and Q_j^2 fixed and transform P_j^1 and P_j^2 by

$$P_j^1 \to P_j^1 - \tfrac{1}{2}F(Q_j^1 - Q_j^2)\,,$$
$$P_j^2 \to P_j^2 + \tfrac{1}{2}F(Q_j^1 - Q_j^2)\,,$$

we generate a canonical transformation which causes H_1 and H_2 to undergo the transformation

$$H_1 \to [(\mathbf{P}^1)^2 + m_1^2]^{1/2}$$
$$H_2 \to [(\mathbf{P}^2)^2 + m_2^2]^{1/2}\,.$$

In fact, this canonical transformation leaves the forms (4.2) of P_j and (4.4) of J_j unchanged,[30] so that it puts the generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ all into the standard "free-particle" forms. We once again note that we have not changed the functions Q_j^1 and Q_j^2 which we have assumed to represent the positions of the particles. (If one identifies the P_j^n with the particle momenta and requires that they transform under Lorentz transformations to uniformly moving reference frames in the manner that is familiar for momenta, one can deduce, without doing any canonical transformations, that the generators have the standard "free-particle" forms; see Appendix D. By using canonical transformations we have arrived at the result from a weaker assumption.) Thus, we can conclude, in particular, that

$$[[Q_j^n,H],H] = 0$$

for $n = 1,2; j = 1,2,3$, which we could also have computed directly without making the canonical transformation. This last equation tells us that both of the particles move with a constant velocity, that is, essentially as free particles.

V. SUMMARY

We have seen that relativistic invariance may involve two different theoretical postulates: symmetry of the theory under the relativistic transformation group, reflecting the invariance of physical laws under changes of reference frame, and explicit transformation properties or manifest invariance of certain quantities. We have introduced a Lie group formalism, for classical or quantum mechanics of particle variables or fields, and have used it to show how symmetry under the Lorentz (or Galilei) group is provided by ten generators satisfying the characteristic Lie bracket equations.

For a classical mechanical theory of a fixed number of particles, the Lorentz transformation formula was assumed for the coordinates of the space–time events that comprise the world lines of the particles as defined by their positions as a function of time. This assumption of manifest invariance was expressed in terms of equations involving the Poisson brackets of the canonical position coordinates with the generators of the Lorentz group. For a theory of two particles, it was shown that the only generators satisfying these latter equations plus the Poisson bracket equations characteristic of Lorentz symmetry are those descriptive of free-particle motion; the combined assumptions of Lorentz symmetry and Lorentz transformation of particle positions rule out any interaction.

[30] This is in fact the same kind of canonical transformation as was used to put the generators $\mathbf{J}$ in the standard form (4.4) since $(1/2)\mathbf{q}F = \nabla W$ with $4W$ equal to the integral of F with respect to $(\mathbf{q})^2$. That the generators $\mathbf{P}$ are unchanged is obvious. That the generators $\mathbf{J}$ are unchanged can be seen from the fact that $\mathbf{q} \times \mathbf{q}F = 0$.

APPENDIX A

We outline here the usual Lorentz or Galilean symmetric description of N noninteracting spinless particles in terms of the ten generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ as functions of the canonical variables Q_j^n and P_j^n, $n=1,2,\cdots N$; $j=1,2,3$. For the Lorentz case the generators are

$$\begin{aligned} H &= \textstyle\sum_n [(\mathbf{P}^n)^2 + m_n^2]^{1/2} \\ \mathbf{P} &= \textstyle\sum_n \mathbf{P}^n \\ \mathbf{J} &= \textstyle\sum_n \mathbf{Q}^n \times \mathbf{P}^n \\ \mathbf{K} &= \textstyle\sum_n \frac{1}{2} \{\mathbf{Q}^n[(\mathbf{P}^n)^2 + m_n^2]^{1/2} + [(\mathbf{P}^n)^2 + m_n^2]^{1/2}\mathbf{Q}^n\}\,, \end{aligned} \tag{A1}$$

where the summation is from $n = 1$ to N. For the Galilei case $\mathbf{P}$ and $\mathbf{J}$ are the same as above but

$$\begin{aligned} H &= \textstyle\sum_n (1/2m_n)(\mathbf{P}^n)^2 \\ \mathbf{K} &= \textstyle\sum_n m_n \mathbf{Q}^n\,. \end{aligned} \tag{A2}$$

The last of Eqs. (A1) is written in a symmetrized form so that these generators can be either real functions in a classical mechanical theory or operators in a quantum mechanical theory. In either case one can easily verify that they satisfy the Lie bracket equations (2.10) and (2.11) or (2.12) characteristic of the Lorentz or Galilei group, respectively.

Let us limit ourselves for the moment to a consideration of only classical mechanics so that the generators are real functions of the canonical variables and the Lie bracket is the Poisson bracket. Then we can easily verify that the generators (A1) or (A2) satisfy Eqs. (3.11), (3.12), and (3.18) or (3.14) characteristic of the transformation properties of a particle position Q_j^n for the Lorentz or Galilei case, respectively. In fact, we can write down explicitly the functions that are images of Q_j^n under the canonical transformations generated by H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$. For both the Lorentz and Galilei case we have that

$$\begin{aligned} e^{[P_k]s}(Q_j^n) &= Q_j^n + \delta_{jk}s \\ e^{[J_3]w}(Q_1^n) &= Q_1^n \cos w - Q_2^n \sin w \\ e^{[J_3]w}(Q_2^n) &= Q_2^n \cos w + Q_1^n \sin w \\ e^{[J_3]w}(Q_3^n) &= Q_3^n \end{aligned} \tag{A3}$$

with similar equations for J_1 and J_2. These exhibit the familiar transformations of a position coordinate under space translation and rotation. For the Lorentz case we find that

$$e^{[H]t}(Q_j^n) = Q_j^n + (1/H_n)P_j^n t \tag{A4}$$

where $H_n = [(\mathbf{P}^n)^2 + m_n^2]^{1/2}$, while for the Galilei case we find that

$$e^{[H]t}(Q_j^n) = Q_j^n + (1/m_n)P_j^n t \tag{A5}$$

showing that the velocities of the particles are independent of the time parameter t and are represented by the functions $(1/H_n)P_j^n$ and $(1/m_n)P_j^n$, respectively. For the Galilei case the equation

$$e^{[K_j]a}(Q_k^n) = Q_k^n \tag{A6}$$

leads to the usual Galilean transformation of the position coordinates as a function of time, as is shown in Sec. III. For the Lorentz case we find that

$$e^{[K_j]a}(Q_k^n) = Q_k^n + (1/H_n)Q_j^n P_k^n[\coth a - (1/H_n)P_j^n]^{-1} \tag{A7}$$

for $i \neq j$, and

$$e^{[K_j]a}(Q_j^n) = Q_j^n[\cosh a - (1/H_n)P_j^n \sinh a]^{-1}. \tag{A8}$$

By using these equations we can easily show that

$$\begin{aligned} e^{[K_j]a}[Q_j^n + (1/H_n)P_j^n t'] &= [Q_j^n + (1/H_n) \\ &\times P_j^n(t' + Q_j^n \sinh a)(\cosh a - (1/H_n)P_j^n \sinh a)^{-1}] \\ &\times \cosh a - (t' + Q_j^n \sinh a) \\ &\times [\cosh a - (1/H_n)P_j^n \sinh a]^{-1} \sinh a \end{aligned}$$

which, in view of Eq. (A4), is equivalent to

$$e^{[K_j]a}(e^{[H]t'}(Q_j^n)) = e^{[H]T_n}(Q_j^n) \cosh a - T_n \sinh a \tag{A9}$$

in terms of the function T_n defined by

$$T_n = (t' + Q_j^n \sinh a)[\cosh a - (1/H_n)P_j^n \sinh a]^{-1}.$$

[Here the expression $e^{[H]T_n}$, where T_n is a function of phase space, is defined by

$$e^{[H]T_n}(A) = A + T_n[A,H] + (1/2!)T_n^2[[A,H],H] + \cdots$$

and Eq. (A4) remains valid if t is replaced by T_n.] In terms of T_n, again using Eq. (A4), we can solve for

$$t' = T_n \cosh a - e^{[H]T_n}(Q_j^n) \sinh a. \tag{A10}$$

Using the same function T_n we can similarly derive the equation

$$e^{[K_j]a}(e^{[H]t'}(Q_k^n)) = e^{[H]T_n}(Q_k^n) \tag{A11}$$

for $j \neq k$. In terms of the parameter $v = \tanh a$, Eqs. (A9) and (A11) take the form

$$e^{[K_j]a}(e^{[H]t'}(Q_j^n)) = (1 - v^2)^{-1/2}[e^{[H]T_n}(Q_j^n) - vT_n]$$
$$t' = (1 - v^2)^{-1/2}[T_n - ve^{[H]T_n}(Q_j^n)]$$

which, together with Eq. (A11), lead, after integration with a delta-function density distribution, to the usual Lorentz transformations of the position coordinates. Note that, here, T_n is not just a parameter but is a function of the canonical variables. For this reason we were very careful to introduce it in such a way that it was not subject to any canonical transformations. We were able to do this because we knew the result (A4) for $e^{[H]t}(Q_j^n)$ as an explicit simple function of t. Our lack of this knowledge in the general case prevents us from exhibiting the Lorentz transformations of the position coordinates in such a simple way and forces us to use other methods such as the expansion to first order in the group parameter in Sec. III.

Let us now briefly consider the Lorentz transformation of the average value of a particle position as a function of time in a Lorentz symmetric quantum mechanical theory described by the generators (A1) for N noninteracting particles. This question is discussed in general in the last part of Sec. III, where it is shown that for a given pure state the average values of the particle positions as a function of time transform according to the usual Lorentz transformation formula if and only if Eq. (3.19) is satisfied by the expectation values for that pure state. For the case of the noninteracting generators (A1), we have

$$[[Q_j^n,H],H] = 0$$

and

$$[Q_j^n,K_k] = \tfrac{1}{2}(Q_k^n[Q_j^n,H] + [Q_j^n,H]Q_k^n),$$

which is the quantum-mechanical analog of Eq. (3.18). If we use these equations to expand Eq. (3.19) for a given pure state as a power series in t' and equate the terms to first order in t', we conclude that we must have

$$\langle([Q_j^n,H])^2\rangle = \langle[Q_j^n,H]\rangle^2$$

for that pure state if Eq. (3.19) is to be valid. This means that the pure state in question must be an "eigenstate" of $[Q_j^n, H]$. Conversely, we can see that whenever the pure state is an "eigenstate" of $[Q_j^n, H]$, Eq. (3.19) will be true. Thus, for the noninteracting generators (A1), a necessary and sufficient condition for the average values of the particle positions as a function of time to transform according to the usual Lorentz transformation formula is that the averages be taken for a state which is an "eigenstate" of the operators $[Q_j^n, H]$.

APPENDIX B

We have collected in a formal outline some basic mathematical ideas involved in constructing a realization of the relativistic transformation group as a Lie group G of automorphisms of the linear space R generated by the Lie algebra L (Sec. II). Particular attention is given to the role of neutral elements in the Lie algebra and the statements made about them in the text are demonstrated here.

Since we do not intend to attain any mathematical rigor, we abandon the more elegant and mathematically complete modern methods of treating the relation between Lie algebras and Lie groups[12,13] and instead use a few directly applicable formal results from the mathematics that was developed during the first decade of this century. If T is an element of L, let $e^{[T]}$ be the automorphism of R defined by the exponential series [Eq. (2.4)]

$$e^{[T]}(F) = F + [F,T] + \tfrac{1}{2}!\,[[F,T],T] + \cdots \tag{B1}$$

for every element F of R. (Recall that the Lie algebra L is a subspace of R and that the Lie bracket is defined for all elements of R.) Then according to the Baker–Campbell–Hausdorff identity,[31–33] if T and V are any two elements of L then

$$e^{[T]}(e^{[V]}(F)) = e^{[W]}(F) \tag{B2}$$

where W is an element of L that can be calculated as a power series in repeated Lie brackets of T and V. The leading terms in this series are

$$W = T + V - \tfrac{1}{2}[T,V] + \tfrac{1}{12}[[T,V],V] + \tfrac{1}{12}[[V,T],T] + \cdots \tag{B3}$$

to terms involving two Lie brackets. The higher order terms are also rational multiples of multiple Lie brackets of T and V. Since real multiples, sums, and Lie brackets of elements of L are contained in L, it is clear that W as given by (B3) is in L, and from this and (B2) it is clear that the set G of automorphisms (B1) of R corresponding to all T belonging to L constitutes a group. As stated, this is all completely formal and its rigorous meaning depends on the convergence of the series. But it has in fact been shown by Dynkin[34] that this method can be used to provide a rigorous connection between the Lie algebra and the local Lie group.

We define a "neutral element" of R to be an element that has a vanishing Lie bracket with every element of R. If C is a neutral element and T is any element of R, then $T + C$ generates the same automorphism (B1) of R as T, for T occurs only inside of Lie brackets. Hence, we are interested in the specification of the generators of the group only up to the addition of a neutral element.

The structure of the Lie group G is determined by the Lie bracket equations, such as Eqs. (2.10) and (2.11) or (2.12) for the Lorentz or Galilei groups, that are satisfied by a basic set of generators in L. But to specify the group of automorphisms these equations need to hold only up to the addition of a neutral element. Suppose that the Lie brackets of some pairs of generators are changed by adding neutral elements to them. Then for any two elements T and V of L, the element W of L calculated by the series (B3) will be changed by at most an added neutral element, so the automorphism generated by W and the composition law (B2) of the group will be unchanged. The group G of automorphisms remains the same if the generators or their Lie brackets are changed by the addition of neutral elements.

If we are to assume that the generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ of the Lorentz or Galilei group satisfy Eqs. (2.10) and (2.11) or (2.12), respectively, we need to show that if they satisfy equations differing from these by added neutral elements they can always be changed, just by adding neutral elements, to generators that satisfy Eqs. (2.10) and (2.11) or (2.12) exactly. This can be done quite simply by using the Jacobi identity to put all of the unwanted neutral elements inside of Lie brackets where their effect is the same as that of the zero element. Thus, if Eqs. (2.10) are true only up to added neutral elements we still have, for example, that

$$\begin{aligned}[P_2,P_3] &= [[J_3,P_1],P_3] \\ &= [[P_3,P_1],J_3] + [[J_3,P_3],P_1] = 0\end{aligned}$$

and we can conclude that

$$[P_j,P_k] = 0 \tag{B4}$$

for j, $k = 1, 2, 3$. In the same way we can prove that

$$[P_j,H] = 0 \tag{B5}$$

and that

$$[J_j,H] = 0 \tag{B6}$$

for $j = 1, 2, 3$. We can always secure the equations

$$[J_i,J_j] = \epsilon_{ijk}J_k \tag{B7}$$

by adding any neutral elements that may occur on the right-hand sides to the three generators J_k. For the properties of the Lie bracket imply that these

[31] H. F. Baker, Proc. London Math. Soc. **34**, 91 (1901); **3**, 24 (1905).
[32] J. E. Campbell, Proc. London Math. Soc. **33**, 285 (1901); *Introductory Treatise on Lie's Theory of Finite Continuous Transformation Groups* (Clarendon Press, Oxford, England, 1903).
[33] F. Hausdorff, Ber. Verh. Saechs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **58**, 19 (1906).
[34] E. B. Dynkin, Am. Math. Soc. Trans. No. 97 (1953).

neutral elements must form an antisymmetric tensor in their dependence on the indices i and j. Now from the equation

$$\begin{aligned}[J_3,P_3] &= [[J_1,J_2],P_3] \\ &= [[P_3,J_2],J_1] + [[J_1,P_3],J_2] \\ &= [J_1,P_1] + [J_2,P_2]\end{aligned}$$

and the similarly derived equation

$$[J_1,P_1] = [J_2,P_2] + [J_3,P_3]$$

we can deduce that

$$[J_2,P_2] = 0$$

and, in general, we can conclude that

$$[J_j,P_j] = 0 \tag{B8}$$

for $j = 1, 2, 3$. We also have that

$$\begin{aligned}[J_2,P_3] &= [[J_3,J_1],P_3] \\ &= [[J_3,P_3],J_1] + [[P_3,J_1],J_3] = -[J_3,P_2]\end{aligned}$$

and, in general, in view of Eq. (B8), we have that

$$[J_j,P_k] = -[J_k,P_j] \tag{B9}$$

for $j, k = 1, 2, 3$. From Eq. (B9) we see that we can establish the equations

$$[J_i,P_j] = \epsilon_{ijk}P_k \tag{B10}$$

by adding any neutral elements that may occur on the right-hand side to the three generators P_k since these neutral elements must form an antisymmetric tensor in their dependence on the indices i and j. In the same way we can show that the equations

$$[J_i,K_j] = \epsilon_{ijk}K_k \tag{B11}$$

can be secured by adding the appropriate neutral elements to the three generators K_k. Now using Eq. (B10), we have that

$$\begin{aligned}[K_2,H] &= [[J_3,K_1],H] = [[H,K_1],J_3] \\ &\quad + [[J_3,H],K_1] = -[P_1,J_3] = P_2\end{aligned}$$

and, in general, we have that

$$[K_j,H] = P_j \tag{B12}$$

for $j = 1, 2, 3$. Thus, by assuming that Eqs. (2.10) are true only up to added neutral elements we have been able, by a repeated use of the Jacobi identity and the addition of neutral elements to **P**, **J**, and **K**, to prove Eqs. (B4)–(B7) and (B10)–(B.12) which are exactly Eqs. (2.10).

For the case of the Lorentz group, if we assume that Eqs. (2.11) are valid only to added neutral elements, we have, using Eq. (B.7), that

$$\begin{aligned}[K_2,K_3] &= [[J_3,K_1],K_3] = [[J_3,K_3],K_1] \\ &\quad + [[K_3,K_1],J_3] = -[J_2,J_3] = -J_1\end{aligned}$$

and, in general, we have that

$$[K_i,K_j] = -\epsilon_{ijk}J_k \tag{B13}$$

for $i, j = 1, 2, 3$. If for the case of the Galilei group we assume that Eqs. (2.12) are valid only to added neutral elements, we can prove in the same way that

$$[K_i,K_j] = 0 \tag{B14}$$

for $i, j = 1, 2, 3$. For either the Lorentz or Galilei case, that is assuming either Eqs. (2.11) or (2.12) to be valid up to added neutral elements, we have that

$$\begin{aligned}[P_2,K_3] &= [[J_3,P_1],K_3] \\ &= [[K_3,P_1],J_3] + [[J_3,K_3],P_1] = 0\end{aligned}$$

and, in general, we have that

$$[P_j,K_k] = 0 \tag{B15}$$

for $j \neq k$. We also have that

$$\begin{aligned}[K_3,P_3] &= [K_3,[J_1,P_2]] = [P_2,[J_1,K_3]] \\ &\quad + [J_1,[K_3,P_2]] = [K_2,P_2]\end{aligned}$$

and, in fact, we can conclude that

$$[K_1,P_1] = [K_2,P_2] = [K_3,P_3]\,. \tag{B16}$$

For the Lorentz case, assuming Eqs. (2.11) to be valid up to added neutral elements and using Eqs. (B.15) and (B.16), we can establish the equations

$$[K_j,P_k] = \delta_{jk}H \tag{B17}$$

by adding the neutral element that may occur on the right-hand side to the generator H. But for the case of the Galilei group, assuming Eqs. (2.12) to be valid up to added neutral elements and again using Eqs. (B.15) and (B.16), we can only conclude that

$$[K_j,P_k] = \delta_{jk}M \tag{B18}$$

where M is a neutral element upon which we can place no restriction. From the assumption of Eqs. (2.11) only up to added neutral elements we have thus been able, by the use of the Jacobi identity and the addition of a neutral element to H, to prove Eqs. (B13) and (B17) which are exactly Eqs. (2.11), and from the assumption of Eqs. (2.12) only up to added neutral elements we have been able to prove Eqs. (B.14) and (B.18) which are exactly Eqs. (2.12). This justifies our assuming Eqs. (2.10) and (2.11) or (2.12) as characteristic of the Lie algebra of the Lorentz or Galilei groups, respectively, and our ignoring of all neutral elements except the M in Eq. (B18). These results are familiar and correspond to the well-known fact that any representation up to a factor of the Lorentz group can be reduced by a proper choice of phase factors to a true representation but that there are representations up to a factor of the Galilei group that can not be so reduced to true representations.[1,35]

APPENDIX C

We continue, in the spirit of the preceding appendix, to give a formal proof of the identity (2.13)

$$e^{[T]s}(e^{[V]r}(F)) = e^{[\exp([T]s)(V)]r}(e^{[T]s}(F))$$

for any generators T and V in L and any element F of R. If we expand both sides of Eq. (2.13) in a power series of the type (2.4) in the parameter r and equate the terms of the first order in r, we find that

$$e^{[T]s}([F,V]) = [e^{[T]s}(F),e^{[T]s}(V)] . \qquad \text{(C1)}$$

Equation (C1), which expresses the preservation of the Lie bracket by the one-parameter group of automorphisms generated by T, follows as a corollary to the identity (2.13). We first prove this corollary (which is also used in the text) and then use it as a lemma to prove the identity (2.13).

From the linearity of the Lie bracket it follows that

$$(d/ds)[e^{[T]s}(F),e^{[T]s}(V)] = [(d/ds)e^{[T]s}(F),e^{[T]s}(V)] + [e^{[T]s}(F),(d/ds)e^{[T]s}(V)]$$
$$= \{[e^{[T]s}(F),T],e^{[T]s}(V)] + [e^{[T]s}(F),[e^{[T]s}(V),T]\}$$

which, by use of the Jacobi identity, can be put in the form

$$(d/ds)[e^{[T]s}(F),e^{[T]s}(V)] = \{[e^{[T]s}(F),e^{[T]s}(V)],T\} .$$

But the left-hand side of Eq. (C1) satisfies the same differential equation

$$(d/ds)e^{[T]s}([F,V]) = [e^{[T]s}([F,V]),T] .$$

Since both sides of Eq. (C1) satisfy the same first-order linear differential equation and are equal at $s = 0$, Eq. (C1) must hold for all s.

Now we can use Eq. (C1) to write

$$(d/dr)e^{[T]s}(e^{[V]r}(F)) = e^{[T]s}([e^{[V]r}(F),V]) = [e^{[T]s}(e^{[V]r}(F)),e^{[T]s}(V)] .$$

We also have that

$$(d/dr)e^{[\exp([T]s)(V)]r}(e^{[T]s}(F)) = [e^{[\exp([T]s)(V)]r}(e^{[T]s}(F)),e^{[T]s}(V)] .$$

Hence, both sides of the identity (2.13) satisfy the same first-order linear differential equation. Since Eq. (2.13) is clearly identically true at $r = 0$, it must then be true for all r.[36]

APPENDIX D

In Sec. III we derived equations characteristic of the transformation properties of a particle position and in Sec. IV we applied these equations, together with the Lie bracket equations characteristic of the relativistic transformation group, to determine the possible relativistically invariant motions of two particles that can be described by our formalism. These equations involve only the functions Q_j^n and their Lie brackets with the generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$, and are based on the identification of the functions Q_j^n with the particle position coordinates. We show here what happens if one also identifies the functions P_j^n with the particle momentum coordinates and requires that they also have the familiar transformation properties. The result is that the Eisenbud–Wigner[26] criterion (see Sec. IV) is established for the case of Galilean invariance, while in the Lorentz case essentially the same conclusion of Sec. IV is again obtained. The main difference for the Lorentz case is that the canonical transformations used in Sec. IV to put the generators in the standard "free-particle" forms are no longer necessary or available. The transformation properties of the P_j^n simply require that the generators have these forms. In other words, for the Lorentz case the transformation properties of the Q_j^n alone imply that the theory is canonically equivalent (from the point of view followed in the text) to the one in which the P_j^n also have the familiar transformation properties.

If we require that $\mathbf{P}^n$ be unchanged by space translations and transform as a vector under space rotations, as is familiar for a momentum, we must have that

$$[P_j^n,P_k] = 0$$
$$[J_i,P_j^n] = \epsilon_{ijk}P_k^n$$

for i, j, $k = 1, 2, 3$. These equations, together with Eqs. (3.11) and (3.12), imply that the generators $\mathbf{P}$ and $\mathbf{J}$ must have the standard "free-particle" forms (4.2) and (4.4) up to an added constant term. For any additional term must have a vanishing Poisson bracket with Q_j^n and with P_j^n. This is the same result that was obtained in Sec. IV, but here it occurs as a necessary consequence of our assumptions instead of

[35] V. Bargmann, Ann. Math. 59, 1 (1954).

[36] The authors are grateful to N. Mukunda for his help in simplifying the formal proofs in this appendix.

through the choice of a suitable canonical transformation of the P_j^n.

Under a transformation to a reference frame moving with a uniform velocity v in the j direction, we require P_k^n to change, in the Galilean case, to

$$e^{[K_j]v}(P_k^n) = P_k^n - \delta_{kj} m_n v$$

as is familiar for a particle momentum under a Galilei transformation. This is equivalent to the equation

$$[K_j, P_k^n] = \delta_{kj} m_n$$

which together with Eq. (3.14) implies that **K** has the standard "free-particle" form, or, in the notation of Eq. (4.6), that $W_j = 0$ except for a possible constant term. From this the "Galilean invariance" criterion of Eisenbud and Wigner[26] follows immediately. This is a result that is established in Sec. IV only on the basis of an assumption about the center-of-mass motion in addition to the Galilean symmetry exhibited in the generators and in the transformation properties of the Q_j^n.

In the Lorentz case, once the generators **P** and **J** have been put in the standard forms (4.2) and (4.4), the treatment in Sec. IV continues unaltered with the P_j^n identified with the particle momenta to the point where H and **K** have been shown to have the forms (4.12) and (4.13) with H_1 and H_2 given by Eqs. (4.18). Now, instead of doing a canonical transformation of the P_j^n to put H_1 and H_2 in the standard free-particle forms, we require that the P_j^n transform as is familiar for particle momenta under Lorentz transformations. From the Lorentz transformation formula

$$p_k'^n(t') = p_k^n(t)$$

for the y or z components of the particle momenta under a Lorentz transformation to a frame moving uniformly with a velocity $v = \tanh s$ in the x direction, and from Eq. (3.15) for t', we can follow an argument which is exactly analogous to that leading from Eqs. (3.15) to Eq. (3.18) to derive that

$$[K_j, P_k^n] = Q_j^n[H, P_k^n]$$

for $j \neq k$. But we can explicitly evaluate the Poisson brackets

$$[K_j, P_k'] = Q_j'(\partial H_1/\partial q_k) + Q_j^2(\partial H_2/\partial q_k)$$
$$Q_j'[H, P_k'] = Q_j'(\partial H_1/\partial q_k + \partial H_2/\partial q_k)$$

by using the variables (4.3) and the known forms of H and **K**. Since these two expressions must be equal we have that

$$q_j(\partial H_2/\partial q_k) = 0$$

for $j \neq k$ which means that F must vanish and H_1 and H_2 must have the standard "free-particle" forms. Again we have reached, as a necessary consequence of our assumptions, the same conclusion as is reached in Sec. IV by making a suitable canonical transformation of the P_j^n. For the Lorentz case the identification of the P_j^n with the particle momenta, together with the requirement that they transform in the familiar manner, serve only to remove an arbitrariness of representation by picking one set of generators from a class of sets of generators which, from the point of view followed in the text, are all canonically equivalent by transformations of the P_j^n.