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# Electromagnetic Properties of Stable Particles and Resonances According to the Unitary Symmetry Theory (*).

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**Summary.** — We show that a theory whose $SU_3$-invariant strong interactions are perturbed by electromagnetic interactions alone may be obtained formally by a certain unitary transformation of a theory whose $SU_3$-invariant strong interactions are perturbed by merely charge-independent interactions. By exploiting readily available information on the latter theory, we can give a rapid derivation of relationships between the electromagnetic properties of various particles and resonances. While many new results are presented, principal emphasis is on the power of the method and on the recognition of the precise nature of the assumptions necessary for the derivation of results. For example, of the seven familiar relations between the magnetic moments of baryons only two require use of the assumption that magnetic moments depend linearly on the electromagnetic charge-current density; the others exemplify relations which hold in identical form for all electromagnetic properties of baryons — electric and magnetic form factors, electromagnetic self-energies, Compton scattering amplitudes etc.

## 1. – Introduction.

At the present time, the higher scheme that has achieved most success in connection with the recently discovered baryon and meson resonances is the unitary symmetry theory of NE'EMAN (1) and GELL-MANN (2) which is

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(1) Y. NE'EMAN: *Nucl. Phys.*, **26**, 222 (1961).
(2) M. GELL-MANN: Cal. Tech. Report CTSL-20 (1961), unpublished, and *Phys. Rev.*, **125**, 1067 (1962).

based on the group $SU$. Predictions of the theory so far investigated have related not only to strong interactions but also to electromagnetic, and even to a limited extent, to weak interactions. In many of the previous investigations, however, the methods of computation used have been suited specifically to situations involving only low- (eight and ten) dimensional irreducible representation (IR's) of $SU_3$ and even then lead to the expenditure of much arithmetical effort. Evidently, if higher IR's of $SU_3$ become important or if more complex physical situations are to be discussed, it is essential that general and readily applicable computational techniques be developed. Elsewhere [3], with this end in view, we have examined the behavior of basic states of IR's of $SU_3$ under Weyl reflections or « generalized charge-symmetry operations » [4]. In the present paper we make an application to the derivation of the electromagnetic properties of various unitary multiplets [5].

Emphasis in our work is placed on the simplicity and generality of the method employed. The basic idea is to exploit the relationship of the transformation properties with respect to $SU_3$ of the electromagnetic charge-current density and of the (postulated) term of the strong interaction Hamiltonian, which breaks its exact invariance with respect to $SU_3$ and produces the mass splitting of unitary multiplets into isotopic multiplets. The two quantities in fact both transform like components of the tensor operator associated with the regular representation—the eight component IR (1, 1)—of $SU_3$, and can be transformed the one into the other by a suitable Weyl reflection. The actual reflection is that which is denoted by $W_3$ in MSD; we note that it is an element of $SU_3$. It follows then that a theory (first theory) whose $SU_3$-invariant strong interactions are perturbed by electromagnetic interactions alone may be formally obtained by a unitary transformation (generated by $W_3$) of a theory (second theory) whose $SU_3$-invariant strong interactions are perturbed by charge-independent interactions. By exploiting readily available information on the second theory, we can give a rapid derivation of relationship between the electromagnetic properties of the components of unitary multiplets. Amongst our results, we distinguish two categories of predictions, which we call electromagnetic relations of the first and of the second kind. The significance of and distinction between the two categories is explained in the following paragraphs.

Equalities between the masses of different members of any isotopic sub-

(3) A. J. MACFARLANE, E. C. G. SUDARSHAN and C. DULLEMOND: *Nuovo Cimento*, **30**, 845 (1963). Hereafter we refer to this paper as MSD.

(4) Y. YAMAGUCHI: *Prog. Theor. Phys. Suppl.*, **11**, 1 (1960). P. T. MATTHEWS and A. SALAM: *Proc. Phys. Soc.*, **80**, 28 (1962).

(5) By unitary multiplet, we mean a set of particles or resonances associated with the basis states of some IR of $SU_3$.

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multiplet in the second theory hold to all orders in the charge-independent perturbations. They correspond to equalities in the first theory between the electromagnetic properties of different members of any $W_3$-transformed isotopic submultiplet of a unitary multiplet. Similarly vanishing of transition masses between members of different isotopic submultiplets of a unitary multiplet in the first theory corresponds to the forbiddenness of electromagnetic transitions between members of different $W_3$-transformed isotopic submultiplets of a unitary multiplet in the second theory. In each case, translation of statements for electromagnetic properties of $W_3$-transformed states in terms of original basis states leads to electromagnetic relations, which we say are of the first kind.

Corresponding to the various mass formulae which in the second theory relate the common mass values of different isotopic submultiplets of a unitary multiplet, we obtain, in the second theory, linear relationships between the common values, for certain electromagnetic properties, of different $W_3$-transformed isotopic submultiplets of a unitary multiplet. These translate into electromagnetic relations of the second kind.

It is seen that relations of the first kind hold good independently of the order of electromagnetic interactions involved. Accordingly, for a given unitary multiplet, formally identical such relations hold for all electromagnetic quantities. Relations of the second kind are then those which hold in addition when the order of the electromagnetic interactions is specified. Different relations of the second kind consequently are obtained for quantities which depend on electromagnetic interactions to different order. For example, electric and magnetic form factors depend linearly on the electromagnetic interactions, so that for them we obtain, in addition to relations of the first kind, relations of the second kind that follow from a $W_3$-transformation of the first-order Okubo (6) mass formula; whereas, for self energies and Compton scattering amplitudes, relations of the second kind emerge after the use of the corresponding (7) second-order formula.

Some of the results which we derive below—particularly those relating to the baryon octet, which belongs to the IR $(1, 1)$ of $SU_3$—have already been derived by other authors (6,8,9), by methods not always suitable for application to more complex situations. More in the spirit of the present work, is the

(6) S. OKUBO: *Prog. Theor. Phys.*, **27**, 949 (1962).

(7) S. OKUBO: *Phys. Lett.*, **4**, 14 (1963).

(8) S. COLEMAN and S. L. GLASHOW: *Phys. Rev. Lett.*, **6**, 423 (1961); N. CABIBBO and R. GATTO: *Nuovo Cimento*, **21**, 872 (1961).

(9) H. RUEGG: *Nuovo Cimento*, **24**, 461 (1962); R. GATTO: *Seminar in Theoretical Physics* (Trieste, July, 1962), (International Atomic Energy Agency, Vienna, 1963).

approach of LEVINSON, LIPKIN and MESHKOV (10), wherein, the notion of « $U$-spin » is applied to electromagnetic matrix elements. In a sense, the present work can be regarded as a systematic application of the « $U$-spin » formalism to electromagnetic quantities, since what we have called $W_3$-transformed isotopic multiplets are essentially the same as their $U$-multiplets. However, they have been concerned principally with photoproduction matrix elements and have usually only selected conspicuous results to illustrate their ideas while we have obtained complete sets of results in all cases considered. It is probably worth stressing at this point, in view of the important role such states play in our work, that the results from MSD which we use for the $W_3$-transformed basis-states of an IR of $SU_3$, give not only correct linear combinations of the original basis states in cases where nonsimple weights are involved but also, within each IR considered, a consistent set of phases.

The material of the paper is organized in the following way. In Sect. **2**, after a brief survey of relevant facts about the IR's of $SU_3$ and the Weyl reflection $W_3$, we discuss electromagnetic relations of the first kind. Then we go on to predictions of the second kind dealing with form factors in Sect. **3**, and self-energies and masses in Sect. **4**. Some additional comments are offered in a final fifth Section. In general, we have separated our results from the main text of the paper and presented them in a series of tables.

## 2. – Electromagnetic relations of the first kind.

We begin with a brief discussion of notation, the IR's of $SU_3$ and the Weyl reflection $W_3$. The group $SU_3$ is generated by a set of eight operators

$$H_1, \quad H_2, \quad E_{\pm 1}, \quad E_{\pm 2}, \quad E_{\pm 3},$$

which satisfy a standard set of commutation relations, as given by eqs. (II.12), (II.17) and (II.18) of ref. (11). Contact with physics follows from the identi-

(10) C. A. LEVINSON, H. J. LIPKIN and S. MESHKOV: *Unitary Symmetry in Photoproduction and Other Electromagnetic Interactions*, to be published in *Phys. Lett.* We thank these authors for making their work known to us prior to its publication. « After the present work was finished, a paper by S. P. ROSEN: *Phys. Rev. Lett.*, **11**, 100 (1963), appeared using a formalism identical to that of LEVINSON *et al.*, and essentially equivalent to that employed here. It ought also to be pointed out that CABIBBO and GATTO (8) were the first to realize that there was an $SU_2$ subgroup of $SU_3$ that leaves invariant a theory with $SU_3$ invariant interactions plus electromagnetism. »; See also R. J. OAKES: *Phys. Rev.*, **132**, 2344 (1963).

(11) R. E. BEHRENDS, J. DREITLEIN, C. FRONSDAL and B. W. LEE: *Rev. Mod. Phys.*, **34**, 1 (1962).

fications

$$\sqrt{3}\,H_1 \rightarrow I_z\,,$$
$$\sqrt{6}\,E_{\pm 1} \rightarrow I_\pm \quad I_x \pm iI_y$$
$$2H_2 \rightarrow Y$$

For any IR of $SU_3$ we may introduce a basis $|I\nu Y\rangle$ where $I(I+1)$, $\nu$ and $Y$ are the eigenvalues of $\boldsymbol{I}^2$ (total isospin), $I_z$ ($z$ component of isospin) and $Y$ (hypercharge) respectively. For the IR's of $SU_3$, we use the notation $(\lambda, \mu)$ whose significance is explained in Sect. **2** of MSD. We refer to this source for a discussion of the properties of the IR $(\lambda, \mu)$ of $SU_3$, in particular for its allowed weights, their multiplicities and its weight diagram.

We define the Weyl reflections $W_\alpha$ $(\alpha = 1, 2, 3)$ of $SU_3$ to be the operations of reflection respectively in the axis [12] 1, 2 and 3 of the weight space of any IR of $SU_3$. They satisfy [3]

$$\left\{\begin{array}{l} W_\alpha^2 = 1\,, \qquad \alpha = 1, 2, 3\,, \\ \text{and} \\ W_3 = W_1 W_2 W_1 = W_3^\dagger\,, \end{array}\right.$$

and have known [3] commutation relations with the generators (2.1) of $SU_3$. Of the results quoted explicitly in MSD (eq. (3.7)) we here note only the following one

$$W_3 Y W_3 = -(I_z + \tfrac{1}{2}Y) = -Q\,, \tag{2.4}$$

where $Q$ is the electric charge; the others were used in the calculation of the behaviour under $W_3$ of the basis vectors $|I\nu Y\rangle$ of the IR's of $SU_3$. Explicit forms for $W_3\,|I\nu Y\rangle$ for the IR's (1,1), (2,2) and (4,1) appear in MSD eqs. (3.13), (3.26) and (3.28), respectively. In addition to these results, we require also the results for the IR (3,0) which are [13]

$$\left\{\begin{array}{l} W_3\,|\tfrac{3}{2}\;-\tfrac{3}{2}\;1\rangle = |\tfrac{3}{2}\;-\tfrac{3}{2}\;1\rangle\,, \\ W_3\,|\tfrac{3}{2}\;-\tfrac{1}{2}\;1\rangle = |1\;-1\;0\rangle\,, \\ W_3\,|\tfrac{3}{2}\;\;\tfrac{1}{2}\;1\rangle = |\tfrac{1}{2}\;-\tfrac{1}{2}\;-1\rangle\,, \\ W_3\,|\tfrac{3}{2}\;\;\tfrac{3}{2}\;1\rangle = |0\;\;0\;-2\rangle\,, \\ W_3\,|1\;\;1\;0\rangle = |\tfrac{1}{2}\;\tfrac{1}{2}\;-1\rangle\,, \\ W_2\,|1\;0\;0\rangle \qquad |1\;0\;0\rangle\,. \end{array}\right.$$

[12] We have employed the same labelling of axes as is used in ref. [10].

[13] These results were not given in MSD since the IR (3, 0) involves only simple weights so that the effect of Weyl reflections on its basis follow from inspection of its weight diagram.

We are now in a position to describe the central role occupied by $W_3$ in the present approach to electromagnetic properties of unitary multiplets.

Denote by $\mathscr{I}_z$ and $\mathscr{Y}$ the components of the tensor operator associated with the regular representation of $SU_3$ which transform under $SU_3$ like the generators $I_z$ and $Y$ and introduce

$$\mathscr{Q} = \mathscr{I}_z + \tfrac{1}{2}\mathscr{Y}$$

In analogy to (2.4) we have

$$W_3 \mathscr{Y} W_3 = \quad \mathscr{Q}$$

We now assert that any electromagnetic property, depending on some operator function $\Phi(\mathscr{Q})$, has matrix elements in any IR $(\lambda, \mu)$ of $SU_3$ which may be evaluated using (2.7) and the more readily available information regarding the matrix elements of operator functions of $\mathscr{Y}$. If $|\psi_i\rangle$ and $|\psi_f\rangle$ are basis states of some IR $(\lambda, \mu)$ of $SU_3$, we can formulate this assertion explicitly as follows

$$\text{(2.8)} \qquad \langle\psi_f|\Phi(\mathscr{Q})|\psi_i\rangle = \langle\psi_f|\Phi(-W_3\mathscr{Y}W_3)|\psi_i\rangle = \\ = \langle\psi_f|W_3\Phi(-\mathscr{Y})W_3|\psi_i\rangle \quad \langle\chi_f|\Phi(-\mathscr{Y})|\chi_i\rangle$$

where $|\chi_{f,i}\rangle = W_3|\psi_{f,i}\rangle$ are known by virtue of MSD. Now, $\mathscr{Y}$ is diagonal with respect to $I$, $\nu$ and $Y$ and application of the Wigner-Eckart theorem for the isospin subgroup of $SU_3$ tells us that its matrix elements are independent of $\nu$. Thus in terms of an explicit notation for states of $(\lambda, \mu)$, we further develop (2.8) using

$$\text{(2.9)} \qquad \langle\lambda\mu I'\nu' Y'|\Phi(-\mathscr{Y})|\lambda\mu I\nu Y\rangle \quad \varphi(I, Y)\,\delta(II')\,\delta(\nu\nu')\,\delta(YY') \,.$$

In addition to depending as indicated on $I$ and $Y$, the function [14] $\varphi(I, Y)$ depends on $\lambda$, $\mu$ and the actual form of $\Phi(\mathscr{Q})$. We refer to those predictions regarding electromagnetic properties of unitary multiplets which follow from (2.8) and (2.9) without the use of any assumption about the form of $\Phi(\mathscr{Q})$ as electromagnetic properties of the first kind.

Before we illustrate this by calculating the electromagnetic properties of the first kind for the stable baryon octet, we make some comments to clarify the relation of the above formal discussion to the physical description of it given in the introduction. Firstly, since the electromagnetic interaction transforms like $\mathscr{Q}$ with respect to $SU_3$, the mass-splitting charge-independent interaction transforms like $\mathscr{Y}$ with respect to $SU_3$, and $W_3$, being an element of $SU_3$, generates a unitary transformation, we see that (2.8) explicitly relates matrix

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([14]) Normally in what follows we do not explicitly indicate the $\lambda$ and $\mu$ labels of functions like $\varphi(I, Y)$ or of unknown quantities like $B$ and $C$ of (3, 1).

elements in a theory (first theory) whose $SU_3$ invariant strong interactions are perturbed only by electromagnetic interactions to matrix elements in a theory (second theory) whose $SU_3$ invariant strong interactions are perturbed only by charge-independent strong interactions. Secondly, (2.9) states that in the second theory particles of the same isotopic submultiplet of a unitary multiplet have equal masses to all orders in the perturbation, and that transition masses between particles of different isotopic submultiplets vanish also to all orders in the perturbation. As pointed out in the introduction, results that stem from a unitary transformation, generated by $W_3$, of these two statements are termed electromagnetic relation of the first kind. It is important to stress that they involve no assumption about the dependence of $\Phi(\mathscr{Q})$ on $\mathscr{Q}$. As a consequence, formally identical results apply to electric and magnetic form factors, self-energies, Compton scattering amplitudes, double Compton amplitudes, and amplitudes for the annihilation of particle and antiparticle pairs into an arbitrary number of photons, etc.

We now illustrate using the stable baryons which belong to the eight-component IR(1,1) of $SU_3$. From (2.8), (2.9) of the present work and (3.13) of MSD, we obtain the following nonvanishing electromagnetic matrix elements in terms of four unknown functions $\varphi(I, Y)$

$$
(2.10)\qquad \left|\begin{array}{l}
\langle \tfrac12\ \ \tfrac12\ \ 1|\Phi(\mathscr{Q})|\tfrac12\ \ \tfrac12\ \ 1\rangle = \langle 1\ \ 1\ \ 0|\Phi(\mathscr{Q})|1\ \ 1\ \ 0\rangle = \varphi(\tfrac12, -1)\,,\\
\langle \tfrac12\ -\tfrac12\ \ 1|\Phi(\mathscr{Q})|\tfrac12\ -\tfrac12\ \ 1\rangle = \langle \tfrac12\ \ \tfrac12\ -1|\Phi(\mathscr{Q})|\tfrac12\ \ \tfrac12\ -1\rangle = \varphi(1,\ \ 0)\,,\\
\langle \tfrac12\ -\tfrac12\ -1|\Phi(\mathscr{Q})|\tfrac12\ -\tfrac12\ -1\rangle = \langle 1\ -1\ \ 0|\Phi(\mathscr{Q})|1\ \ 1\ \ 0\rangle = \varphi(\tfrac12,\ \ 1)\,,\\
\langle 1\,0\,0|\Phi(\mathscr{Q})|1\,0\,0\rangle = \tfrac14\varphi(1, 0) + \tfrac34\varphi(0, 0)\,,\\
\langle 0\,0\,0|\Phi(\mathscr{Q})|0\,0\,0\rangle = \tfrac34\varphi(1, 0) + \tfrac14\varphi(0, 0)\,,\\
\langle 1\,0\,0|\Phi(\mathscr{Q})|0\,0\,0\rangle = (\sqrt{3}/4)\,[\varphi(1, 0) - \varphi(0, 0)]
\end{array}\right.
$$

We have used time-reversal invariance to equate $\langle 0\ 0\ 0|\Phi(\mathscr{Q})|1\ 0\ 0\rangle$ to $\langle 1\ 0\ 0|\Phi(\mathscr{Q})|0\ 0\ 0\rangle$. Writing $\langle \frac12\,\frac12\,1|\Phi(\mathscr{Q})|\frac12\,\frac12\,1\rangle = \Phi(\mathrm{p})$ etc., we obtain five relationships between nine observable quantities, namely

$$
(2.11)\qquad \left\{\begin{aligned}
\Phi(\mathrm{p}) &= (\Sigma^+)\,,\\
\Phi(\mathrm{n}) &= \Phi(\Xi^0) = \tfrac32\,\Phi(\Lambda) - \tfrac12\,\Phi(\Sigma^0),\\
\Phi(\Sigma^-) &= \Phi(\Xi^-)\,,\\
2\,\Phi_T(\Sigma^0, \Lambda) &= \sqrt{3}\,[\Phi(\Lambda) - \Phi(\Sigma^0)]\,.
\end{aligned}\right.
$$

We have derived electromagnetic relations of the first kind also for other unitary multiplets—(3,0), (2,2) and (4,1)—using the same method, *i.e.* eqs. (2.8) and (2.9) along with eq. (2.5) for (3,0) and eqs. (3.26) and (3.28) of MSD for the respective cases of (2,2) and (4,1). Results are displayed in parametric form in Table I.

TABLE I. – *Electromagnetic matrix elements.*

IR (3, 0)

$$\begin{aligned}
\langle \tfrac{3}{2}\; -\tfrac{3}{2}\quad 1|\Phi|\tfrac{3}{2}\; -\tfrac{3}{2}\quad 1\rangle &= \langle 1\; -1\quad 0|\Phi|1\; -1\quad 0\rangle = \\
&= \langle \tfrac{1}{2}\; -\tfrac{1}{2}\; -1|\Phi|\tfrac{1}{2}\; -\tfrac{1}{2}\; -1\rangle = \\
&= \langle 0\quad 0\; -2|\Phi|0\quad 0\; -2\rangle = \varphi(\tfrac{3}{2}, 1)\\
\langle \tfrac{3}{2}\; -\tfrac{1}{2}\quad 1|\Phi|\tfrac{3}{2}\; -\tfrac{1}{2}\quad 1\rangle &= \langle 1\quad 0\quad 0|\Phi|1\quad 0\quad 0\rangle = \\
&= \langle \tfrac{1}{2}\quad \tfrac{1}{2}\; -1|\Phi|\tfrac{1}{2}\quad \tfrac{1}{2}\; -1\rangle = \varphi(1, 0)\\
\langle \tfrac{3}{2}\quad \tfrac{1}{2}\quad 1|\Phi|\tfrac{3}{2}\quad \tfrac{1}{2}\quad 1\rangle &= \langle 1\quad 1\quad 0|\Phi|1\quad 1\quad 0\rangle = \varphi(\tfrac{1}{2}, -1)\\
\langle \tfrac{3}{2}\quad \tfrac{3}{2}\quad 1|\Phi|\tfrac{3}{2}\quad \tfrac{3}{2}\quad 1\rangle &= \varphi(0, -2)\,.
\end{aligned}$$

IR (2, 2)

$$\begin{aligned}
\langle 2\; -2\quad 0|\Phi|2\; -2\quad 0\rangle &= \langle \tfrac{3}{2}\; -\tfrac{3}{2}\; -1|\Phi|\tfrac{3}{2}\; -\tfrac{3}{2}\; -1\rangle = \\
&= \langle 1\; -1\; -2|\Phi|1\; -1\; -2\rangle = \varphi(1, 2)\\
\langle \tfrac{3}{2}\; -\tfrac{3}{2}\quad 1|\Phi|\tfrac{3}{2}\; -\tfrac{3}{2}\quad 1\rangle &= \langle 1\quad 0\; -2|\Phi|1\quad 0\; -2\rangle = \varphi(\tfrac{3}{2}, 1)\\
\langle 1\; -1\quad 2|\Phi|1\; -1\quad 2\rangle &= \langle 1\quad 1\; -2|\Phi|1\quad 1\; -2\rangle = \varphi(2, 0)\\
\langle 1\quad 0\quad 2|\Phi|1\quad 0\quad 2\rangle &= \langle \tfrac{3}{2}\quad \tfrac{3}{2}\; -1|\Phi|\tfrac{3}{2}\quad \tfrac{3}{2}\; -1\rangle = \varphi(\tfrac{3}{2}, -1)\\
\langle 1\quad 1\quad 2|\Phi|1\quad 1\quad 2\rangle &= \langle \tfrac{3}{2}\quad \tfrac{3}{2}\quad 1|\Phi|\tfrac{3}{2}\quad \tfrac{3}{2}\quad 1\rangle = \\
&= \langle 2\quad 2\quad 0|\Phi|2\quad 2\quad 0\rangle = \varphi(1, -2)\\
\langle 2\; -1\quad 0|\Phi|2\; -1\quad 0\rangle &= \tfrac{1}{6}\,\varphi(\tfrac{3}{2}, 1) + \tfrac{5}{6}\,\varphi(\tfrac{1}{2}, 1)\\
\langle 1\; -1\quad 0|\Phi|1\; -1\quad 0\rangle &= \tfrac{5}{6}\,\varphi(\tfrac{3}{2}, 1) + \tfrac{1}{6}\,\varphi(\tfrac{1}{2}, 1)\\
\langle 2\; -1\quad 0|\Phi|1\; -1\quad 0\rangle &= (\sqrt{5}/6)[\varphi(\tfrac{3}{2}, 1) - \varphi(\tfrac{1}{2}, 1)]\\
\langle \tfrac{3}{2}\; -\tfrac{1}{2}\; -1|\Phi|\tfrac{3}{2}\; -\tfrac{1}{2}\; -1\rangle &= \tfrac{4}{9}\,\varphi(\tfrac{3}{2}, 1) + \tfrac{5}{9}\,\varphi(\tfrac{1}{2}, 1)\\
\langle \tfrac{1}{2}\; -\tfrac{1}{2}\; -1|\Phi|\tfrac{1}{2}\; -\tfrac{1}{2}\; -1\rangle &= \tfrac{5}{9}\,\varphi(\tfrac{3}{2}, 1) + \tfrac{4}{9}\,\varphi(\tfrac{1}{2}, 1)\\
\langle \tfrac{3}{2}\; -\tfrac{1}{2}\; -1|\Phi|\tfrac{1}{2}\; -\tfrac{1}{2}\; -1\rangle &= (2\sqrt{5}/9)[\varphi(\tfrac{3}{2}, 1) - \varphi(\tfrac{1}{2}, 1)]\\
\langle \tfrac{3}{2}\quad \tfrac{1}{2}\; -1|\Phi|\tfrac{3}{2}\quad \tfrac{1}{2}\; -1\rangle &= \langle \tfrac{3}{2}\; -\tfrac{1}{2}\quad 1|\Phi|\tfrac{3}{2}\; -\tfrac{1}{2}\quad 1\rangle = \\
&= \tfrac{1}{6}\varphi(2, 0) + \tfrac{5}{6}\,\varphi(1, 0)\\
\langle \tfrac{1}{2}\quad \tfrac{1}{2}\; -1|\Phi|\tfrac{1}{2}\quad \tfrac{1}{2}\; -1\rangle &= \langle \tfrac{1}{2}\; -\tfrac{1}{2}\quad 1|\Phi|\tfrac{1}{2}\; -\tfrac{1}{2}\quad 1\rangle = \\
&= \tfrac{5}{6}\,\varphi(2, 0) + \tfrac{1}{6}\,\varphi(1, 0)\\
\langle \tfrac{3}{2}\quad \tfrac{1}{2}\; -1|\Phi|\tfrac{1}{2}\quad \tfrac{1}{2}\; -1\rangle &= \langle \tfrac{3}{2}\; -\tfrac{1}{2}\quad 1|\Phi|\tfrac{1}{2}\; -\tfrac{1}{2}\quad 1\rangle = \\
&= (\sqrt{5}/6)[\varphi(2, 0) - \varphi(1, 0)]\\
\langle \tfrac{3}{2}\quad \tfrac{1}{2}\quad 1|\Phi|\tfrac{3}{2}\quad \tfrac{1}{2}\quad 1\rangle &= \tfrac{4}{9}\,\varphi(\tfrac{3}{2}, -1) + \tfrac{5}{9}\,\varphi(\tfrac{1}{2}, -1)\\
\langle \tfrac{1}{2}\quad \tfrac{1}{2}\quad 1|\Phi|\tfrac{1}{2}\quad \tfrac{1}{2}\quad 1\rangle &= \tfrac{5}{9}\,\varphi(\tfrac{3}{2}, -1) + \tfrac{4}{9}\,\varphi(\tfrac{1}{2}, -1)\\
\langle \tfrac{3}{2}\quad \tfrac{1}{2}\quad 1|\Phi|\tfrac{1}{2}\quad \tfrac{1}{2}\quad 1\rangle &= (2\sqrt{5}/9)[\varphi(\tfrac{3}{2}, -1) - \varphi(\tfrac{1}{2}, -1)]\\
\langle 2\quad 1\quad 0|\Phi|2\quad 1\quad 0\rangle &= \tfrac{1}{6}\,\varphi(\tfrac{3}{2}, -1) + \tfrac{5}{6}\,\varphi(\tfrac{1}{2}, -1)\\
\langle 1\quad 1\quad 0|\Phi|1\quad 1\quad 0\rangle &= \tfrac{5}{6}\,\varphi(\tfrac{3}{2}, -1) + \tfrac{1}{6}\,\varphi(\tfrac{1}{2}, -1)\\
\langle 2\quad 1\quad 0|\Phi|1\quad 1\quad 0\rangle &= (\sqrt{5}/6)[\varphi(\tfrac{3}{2}, -1) - \varphi(\tfrac{1}{2}, -1)]\\
\langle 2\quad 0\quad 0|\Phi|2\quad 0\quad 0\rangle &= (1/36)\,\varphi(2, 0) + (15/36)\,\varphi(1, 0) + (20/36)\,\varphi(0, 0)\\
\langle 1\quad 0\quad 0|\Phi|1\quad 0\quad 0\rangle &= (15/36)\,\varphi(2, 0) + (9/36)\,\varphi(1, 0) + (12/36)\,\varphi(0, 0)\\
\langle 0\quad 0\quad 0|\Phi|0\quad 0\quad 0\rangle &= (20/36)\,\varphi(2, 0) + (12/36)\,\varphi(1, 0) + (4/36)\,\varphi(0, 0)\\
\langle 2\quad 0\quad 0|\Phi|1\quad 0\quad 0\rangle &= (\sqrt{15}/36)\,\varphi(2, 0) + (3\sqrt{15}/36)\,\varphi(1, 0) - \\
&\quad - (4\sqrt{15}/36)\,\varphi(0, 0)\\
\langle 1\quad 0\quad 0|\Phi|0\quad 0\quad 0\rangle &= (10\sqrt{3}/36)\,\varphi(2, 0) - (6\sqrt{3}/36)\,\varphi(1, 0) - \\
&\quad - (4\sqrt{3}/36)\,\varphi(0, 0)\\
\langle 0\quad 0\quad 0|\Phi|2\quad 0\quad 0\rangle &= (2\sqrt{5}/36)\,\varphi(2, 0) - (6\sqrt{5}/36)\,\varphi(1, 0) + \\
&\quad + (4\sqrt{5}/36)\,\varphi(0, 0)\,.
\end{aligned}$$

IR (4, 1)

$$\begin{aligned}
\langle \tfrac{5}{2}\; -\tfrac{5}{2}\quad 1|\Phi|\tfrac{5}{2}\quad \tfrac{5}{2}\quad 1\rangle &= \langle 2\; -2\quad 0|\Phi|2\; -2\quad 0\rangle = \\
&= \langle \tfrac{3}{2}\; -\tfrac{3}{2}\quad 1|\Phi|\tfrac{3}{2}\; -\tfrac{3}{2}\quad 1\rangle = \\
&= \langle 1\; -1\quad 2|\Phi|1\; -1\quad 2\rangle = \\
&= \langle \tfrac{1}{2}\; -\tfrac{1}{2}\quad 3|\Phi|\tfrac{1}{2}\; -\tfrac{1}{2}\quad 3\rangle = \varphi(2, 2)
\end{aligned}$$

TABLE 1 (*continued*)

$$
\begin{array}{llllll}
\langle 2\;-2 & 2\,|\Phi|\,2 & 2 & 2\rangle = \langle \tfrac{1}{2}\;\;\tfrac{1}{2}\;-3\,|\Phi|\,\tfrac{1}{2}\;\;\tfrac{1}{2}\;-3\rangle = \varphi(\tfrac{5}{2},1) \\
\langle 2\;-1 & 2\,|\Phi|\,2 & 1 & 2\rangle = \langle 1\;\;1\;-2\,|\Phi|\,1\;\;1\;-2\rangle = \varphi(2,0) \\
\langle 2\;\;0 & 2\,|\Phi|\,2 & 0 & 2\rangle = \langle \tfrac{3}{2}\;\;\tfrac{3}{2}\;-1\,|\Phi|\,\tfrac{3}{2}\;\;\tfrac{3}{2}\;-1\rangle = \varphi(\tfrac{3}{2},1) \\
\langle 2\;\;1 & 2\,|\Phi|\,2 & 1 & 2\rangle = \langle 2\;\;2\;\;0\,|\Phi|\,2\;\;2\;\;0\rangle = \varphi(1,-2) \\
\langle 2\;\;2 & 2\,|\Phi|\,2 & 2 & 2\rangle = \langle \tfrac{5}{2}\;\;\tfrac{5}{2}\;\;1\,|\Phi|\,\tfrac{5}{2}\;\;\tfrac{5}{2}\;\;1\rangle = \varphi(\tfrac{1}{2},-3) \\
\langle 2\;-1 & 0\,|\Phi|\,2 & 1 & 0\rangle = \tfrac{1}{10}\varphi(\tfrac{5}{2},1) + \tfrac{9}{10}\varphi(\tfrac{3}{2},1) \\
\langle 1\;-1 & 0\,|\Phi|\,1 & 1 & 0\rangle = \tfrac{9}{10}\varphi(\tfrac{5}{2},1) + \tfrac{1}{10}\varphi(\tfrac{3}{2},1) \\
\langle 2\;-1 & 0\,|\Phi|\,1 & 1 & 0\rangle = \tfrac{3}{10}[\varphi(\tfrac{5}{2},1) - \varphi(\tfrac{3}{2},1)] \\
\langle \tfrac{5}{2}\;-\tfrac{3}{2} & 1\,|\Phi|\,\tfrac{5}{2} & \tfrac{3}{2} & 1\rangle = \tfrac{1}{25}\varphi(\tfrac{5}{2},1) + \tfrac{24}{25}\varphi(\tfrac{3}{2},1) \\
\langle \tfrac{3}{2}\;-\tfrac{3}{2} & 1\,|\Phi|\,\tfrac{3}{2} & \tfrac{3}{2} & 1\rangle = \tfrac{24}{25}\varphi(\tfrac{5}{2},1) + \tfrac{1}{25}\varphi(\tfrac{3}{2},1) \\
\langle \tfrac{5}{2}\;-\tfrac{3}{2} & 1\,|\Phi|\,\tfrac{3}{2} & \tfrac{3}{2} & 1\rangle = (2\sqrt{6}/25)[\varphi(\tfrac{5}{2},1) - \varphi(\tfrac{3}{2},1)] \\
\langle \tfrac{5}{2}\;-\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{5}{2} & \tfrac{1}{2} & 1\rangle = \tfrac{1}{10}\varphi(2,0) + \tfrac{9}{10}\varphi(1,0) \\
\langle \tfrac{3}{2}\;-\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{3}{2} & \tfrac{1}{2} & 1\rangle = \tfrac{9}{10}\varphi(2,0) + \tfrac{1}{10}\varphi(1,0) \\
\langle \tfrac{5}{2}\;-\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{3}{2} & \tfrac{1}{2} & 1\rangle = \tfrac{3}{10}[\varphi(2,0) - \varphi(1,0)] \\
\langle 2\;\;0 & 0\,|\Phi|\,2 & 0 & 0\rangle = \tfrac{1}{4}\varphi(2,0) + \tfrac{8}{4}\varphi(1,0) \\
\langle 1\;\;0 & 0\,|\Phi|\,1 & 0 & 0\rangle = \tfrac{3}{4}\varphi(2,0) + \tfrac{1}{4}\varphi(1,0) \\
\langle 2\;\;0 & 0\,|\Phi|\,1 & 0 & 0\rangle = (\sqrt{3}/4)[\varphi(2,0) - \varphi(1,0)] \\
\langle \tfrac{5}{2}\;\;\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{5}{2} & \tfrac{1}{2} & 1\rangle = \tfrac{1}{5}\varphi(\tfrac{3}{2},-1) + \tfrac{4}{5}\varphi(\tfrac{1}{2},-1) \\
\langle \tfrac{3}{2}\;\;\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{3}{2} & \tfrac{1}{2} & 1\rangle = \tfrac{4}{5}\varphi(\tfrac{3}{2},-1) + \tfrac{1}{5}\varphi(\tfrac{1}{2},-1) \\
\langle \tfrac{5}{2}\;\;\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{3}{2} & \tfrac{1}{2} & 1\rangle = \tfrac{2}{5}[\varphi(\tfrac{3}{2},-1) - \varphi(\tfrac{1}{2},-1)] \\
 & 0\,|\Phi|\,2 & 1 & 0\rangle = \tfrac{1}{2}\varphi(\tfrac{3}{2},-1) + \tfrac{1}{2}\varphi(\tfrac{1}{2},-1) \\
 & 0\,|\Phi|\,1 & 1 & 0\rangle = \tfrac{1}{2}\varphi(\tfrac{3}{2},-1) + \tfrac{1}{2}\varphi(\tfrac{1}{2},-1) \\
\langle 2\;\;1 & 0\,|\Phi|\,1 & 1 & 0\rangle = \tfrac{1}{2}[\varphi(\tfrac{3}{2},-1) - \varphi(\tfrac{1}{2},-1)] \\
\langle \tfrac{5}{2}\;\;\tfrac{3}{2} & 1\,|\Phi|\,\tfrac{5}{2} & \tfrac{3}{2} & 1\rangle = \tfrac{2}{5}\varphi(1,-2) + \tfrac{3}{5}\varphi(0,-2) \\
 & 1\,|\Phi|\,\tfrac{3}{2} & \tfrac{3}{2} & 1\rangle = \tfrac{3}{5}\varphi(1,-2) + \tfrac{2}{5}\varphi(0,-2) \\
 & 1\,|\Phi|\,\tfrac{3}{2} & \tfrac{3}{2} & 1\rangle = (\sqrt{6}/5)[\varphi(1,-2) - \varphi(0,-2)] \\
\langle 1\;\;0 & 2\,|\Phi|\,1 & 0 & -2\rangle = \tfrac{2}{5}\varphi(\tfrac{5}{2},1) + \tfrac{3}{5}\varphi(\tfrac{3}{2},1) \\
\langle 0\;\;0 & 2\,|\Phi|\,0 & 0 & -2\rangle = \tfrac{3}{5}\varphi(\tfrac{5}{2},1) + \tfrac{2}{5}\varphi(\tfrac{3}{2},1) \\
\langle 1\;\;0 & 2\,|\Phi|\,0 & 0 & -2\rangle = (\sqrt{6}/5)[\varphi(\tfrac{5}{2},1) - \varphi(\tfrac{3}{2},1)] \\
\langle \tfrac{3}{2}\;\;\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{3}{2} & \tfrac{1}{2} & -1\rangle = \tfrac{1}{2}\varphi(2,0) + \tfrac{1}{2}\varphi(1,0) \\
\langle \tfrac{1}{2}\;\;\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{1}{2} & \tfrac{1}{2} & -1\rangle = \tfrac{1}{2}\varphi(2,0) + \tfrac{1}{2}\varphi(1,0) \\
\langle \tfrac{3}{2}\;\;\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{1}{2} & \tfrac{1}{2} & -1\rangle = \tfrac{1}{2}[\varphi(2,0) - \varphi(1,0)] \\
\langle \tfrac{3}{2}\;-\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{3}{2} & \tfrac{1}{2} & -1\rangle = \tfrac{1}{5}\varphi(\tfrac{5}{2},1) + \tfrac{4}{5}\varphi(\tfrac{3}{2},1) \\
\langle \tfrac{1}{2}\;-\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{1}{2} & \tfrac{1}{2} & -1\rangle = \tfrac{4}{5}\varphi(\tfrac{5}{2},1) + \tfrac{1}{5}\varphi(\tfrac{3}{2},1) \\
\langle \tfrac{3}{2}\;-\tfrac{1}{2} & 1\,|\Phi|\,\tfrac{1}{2} & \tfrac{1}{2} & -1\rangle = \tfrac{2}{5}[\varphi(\tfrac{5}{2},1) - \varphi(\tfrac{3}{2},1)] .
\end{array}
$$

## 3. Electromagnetic relations of the second kind: form factors.

We now turn to electromagnetic relations of the second kind which arise as a result of specific assumptions about the dependence of $\Phi(\mathcal{Q})$ on $\mathcal{Q}$. As remarked above, different assumptions are associated with different electromagnetic quantities, giving the relations of the second kind appropriate to these quantities, which augment the relations of the first kind common to all electromagnetic quantities. In this Section, we deal with «linear» electromagnetic quantities such as electric and magnetic form factors, for which the assumption that $\Phi(\mathcal{Q})$ is proportional to $\mathcal{Q}$ involves no essential loss of generality. The use of this form rather than a form $(\alpha+\beta\mathcal{Q})$ for $\mathcal{Q}$ corresponds, physically, to the statement that there is no contribution to the quantities under discussion from nonelectromagnetic interactions, or, put otherwise, to the tracelessness condition used by OKUBO [6]. This is in contrast to the situation wherein we consider the first-order perturbation of exact $SU_3$-invariance by interactions with the transformation properties under $SU_3$ of $\mathcal{Y}$; for there the physical quantity of central interest is mass and the aim is to calculate the induced splittings of the common mass value of a unitary multiplet. The contrast is reflected in the actual form we use for the specialization of $\Phi(I, Y)$ to the present discussion.

In place of (2.9), we have

$$\langle \lambda\mu I\nu Y|(\cdot\ \mathcal{Y})|\lambda\mu I\nu Y\rangle \quad F(I, Y) = BY + C[I(I+1) \quad \tfrac{1}{4}Y^2 \quad \mathcal{C}(\lambda, \mu)].$$

Herein $B$ and $C$ are unknown quantities independent of $I$ and $Y$, and $\mathcal{C}(\lambda, \mu)$ is the eigenvalue [15]

$$\mathcal{C}(\lambda, \mu) \quad [\lambda^2 + \lambda\mu \quad \mu^2 + 3(\lambda + \mu)]/9$$

of the Casimir operator of $SU_3$ in the IR$(\lambda, \mu)$. Result (3.1) of course is simply the Okubo mass formula [6] with the tracelessness condition explicitly built into it. It follows that the linear electromagnetic properties of any unitary multiplet depend on at most two unknown parameters. For the stable baryons, we obtain in an obvious notation

$$
\text{(3.3)}\quad
\begin{array}{lll}
F(\mathrm{n}) = C\,, & F(\mathrm{p}) = -B - \tfrac{1}{2}C\,, & \\
F(\Sigma^-) = B - \quad C\,, & F(\Sigma^0) = -\tfrac{1}{2}C\,, & F(\Sigma^+ \quad B \quad C\,, \\
F(\Lambda) = \tfrac{1}{2}C\,, & F_T(\Sigma^0, \Lambda) = \sqrt{3}/2\,C\,, & \\
F(\Xi^-) = B - \tfrac{1}{2}C\,, & F(\Xi^0) = C &
\end{array}
$$

Thus there are two more relations (of the second kind) for form factors that can be added to those of the first kind already known from (2.11). They are

$$F(\Lambda) + F(\Sigma^0) = 0\,. \tag{3.4}$$

$$F(\Sigma^+) + F(\Sigma^-) \quad 2F(\Sigma^0) \tag{3.5}$$

For the particular case of the magnetic form factors evaluated at zero momentum transfer, *i.e.*, for the anomalous magnetic moments of the baryons

(15) L. C. BIEDENHARN: *Phys. Lett.*, **3**, 69 (1962).

and the mixed moment for the $\Sigma^0 \to \Lambda + \gamma$ decay, eqs. (2.11), (3.4) and (3.5) embody exactly the usual [8,9] complement of results.

For the stable mesons, which also form a unitary octet, the self-charge-conjugate property gives relationships which hold in addition to the analogs of (2.11), (3.4) and (3.5), namely

$$(3.6)\qquad \begin{cases} F(\mathrm{K}^{\cdot} & F(\mathrm{K} & F(\overline{\mathrm{K}}^0) & F(\mathrm{K}^0)\,, \\ F(\pi^{\cdot} & F(\pi^{\cdot} & F(\pi_0) & -(\pi^0)\,, \\ F(\eta) & F(\eta)\,, & F_T(\pi^0, \eta) & \mp F_T(\pi^0, \eta)\,, \end{cases}$$

so that we can express all form factors of the meson octet in terms of pion form factors [8]

$$(3.7)\qquad \begin{cases} F(\mathrm{K}^+ & F(\pi^+) & F(\pi^{\cdot} & F(\mathrm{K}^-)\,, \\ F(\mathrm{K}^0 & F(\overline{\mathrm{K}}^0) & F(\pi^0) \quad F(\eta) & F_T(\pi^0, \eta) \end{cases}$$

In Table II we display relations of the second kind for form factors associated with the unitary multiplets (3,0), (2,2) and (4,1). In the case of the decuplet, we observe that there is really only one parameter involved. This is a general feature that occurs whenever IR's$(\lambda, \mu)$ with $\mu = 0$ (or $\lambda = 0$) are discussed. For the IR$(\lambda, 0)$ we have $I_3 = \frac{1}{2}Y + \frac{1}{3}\lambda$ and, from eqs. (3.1) and (3.2) we get

$$F(I, Y \quad B'Y\,, \qquad B' \quad B + C(\tfrac{1}{2} \quad \tfrac{1}{3}\lambda)$$

When the 27-component IR(2,2) is a self-conjugate meson unitary multiplet, we can, as above in the (1,1) case, augment the results of Table II with results that follow from charge conjugation invariance, *e.g.* the result

$$20\,|\,\Phi(\mathcal{Q})\,|\,2 \quad 20\rangle \qquad \langle 220\,|\,\Phi(\mathcal{Q})\,|\,220$$

implies that the $C$ of eq. (3.1) for this case vanishes.

We may also use a direct approach to linear electromagnetic properties in which matrix elements of $\mathcal{Q}$ aree valuated directly. We start out from the statement that, as far as its matrix elements between states of the same IR of $SU_3$ are concerned $(-\mathcal{Y})$ may be regarded as the following function of the generators of $SU_3$

$$(3.8)\qquad -\mathcal{Y} = BY + C[\boldsymbol{I}^2 - Y^2/4 - \mathscr{C}]\,.$$

Herein $B$ and $C$ are unknown constants and $\mathscr{C}$ is the Casimir operator of $SU_3$, whose eigenvalue for the IR $(\lambda, \mu)$ of $SU_3$ is given by eq. (3.2). With the aid of the work of Sect. 3 of MSD. we mav effect a $W$-transformation of

$\sqrt{15}$

$x($

TABLE II (*continued*).

$$
\begin{array}{ll}
F\left(\tfrac{1}{2} \quad \tfrac{1}{2} \quad 1\right) = B + 35C & F\left(\tfrac{1}{2} - \tfrac{1}{2} - 3\right) = 2B + 10C \\
F\left(\tfrac{1}{2} \quad \tfrac{1}{2} \quad 1\right) = 0\,, & F\left(\tfrac{1}{2} \quad \tfrac{1}{2} - 3\right) = B + 45C \\
F\left(0 \quad 0 \quad 2\right) = B + 25C & \\
F_T(2 \quad 1 \quad 0;\; 1 - 1 \quad 0) = 15C\,, & F_T(\tfrac{5}{2} \quad \tfrac{3}{2} \quad 1;\; \tfrac{3}{2} \quad \tfrac{3}{2} \quad 1) = 4\sqrt{6}\,C \\
F_T(2 \quad 0 \quad 0;\; 1 \quad 0 \quad 0) = 5\sqrt{3}\,C, & F_T(\tfrac{5}{2} \quad \tfrac{1}{2} \quad 1;\; \tfrac{3}{2} \quad \tfrac{1}{2} \quad 1) = 12C \\
F_T(2 \quad 1 \quad 0;\; 1 \quad 1 \quad 0) = 15C, & F_T(\tfrac{5}{2} \quad \tfrac{1}{2} \quad 1;\; \tfrac{3}{2} \quad \tfrac{1}{2} \quad 1) = 12C \\
F_T(1 \quad 0 \quad 2;\; 0 \quad 0 - 2) = 5\sqrt{6}\,C, & F_T(\tfrac{5}{2} \quad \tfrac{3}{2} \quad 1;\; \tfrac{3}{2} \quad \tfrac{3}{2} \quad 1) = 4\sqrt{6}\,C \\
\multicolumn{2}{l}{F_T(\tfrac{3}{2} \quad \tfrac{1}{2} \quad 1;\; \tfrac{1}{2} - \tfrac{1}{2} - 1) = 20C = F_T(\tfrac{3}{2} \quad \tfrac{1}{2} \; -1;\; \tfrac{1}{2} \quad \tfrac{1}{2} \quad 1)}
\end{array}
$$

(3.8), obtaining

$$\mathcal{Q} \quad B'Q \quad C[Q^2 \quad I^- I \quad 6E_{-2}E_2 \quad 2\mathscr{C}]\,, \tag{3.9}$$

with

$$B' \quad B \quad \tfrac{3}{2}C\,,$$

as an operator equation for $\mathcal{Q}$ valid for matrix elements between states of the same IR. Knowing ([15]) explicit forms for the nonvanishing matrix elements, in any IR, of the generators of $SU_3$—they are given in present notation as eqs. (2.13) to (2.16) of MSD—it is a straightforward matter to compute the nonvanishing matrix elements of $\mathcal{Q}$ for any IR of $SU_3$. We note that eq. (3.9) contains terms not necessarily diagonal in I as required and remark that previous results are hereby reproduced.

So far, we have derived electromagnetic relations for particles whose strong interactions are assumed to be exactly invariant with respect to $SU_3$. However, amongst them there are relations which are valid even if the strong interactions of the particles involved are merely charge-independent, *i.e.*, invariant with respect to the isospin subgroup of $SU_3$, and hence valid to all orders in the strong interactions which break exact $SU_3$ invariance. They can be derived using the Wigner-Eckart theorem for the isospin subgroup and the fact that electric charge transforms with respect to it as the sum of an isoscalar and $\nu = 0$ component of an isovector. An example of such a result is the familiar ([16]) eq. (3.5). Other relevant ones are collected in Table III. The basic result used in their derivation is

$$
\begin{aligned}
\langle\lambda\mu I'\nu' Y'|\mathcal{Q}|\lambda\mu I\nu Y\rangle &= \langle\lambda\mu I'\nu' Y \quad (\mathscr{I}_z \quad \tfrac{1}{2}\mathscr{Y})|\lambda\mu I\nu Y\rangle = \\
&= \delta(\nu\nu')\,\delta(YY')\big[\nu a_0(I, \; Y)\,\delta II' \quad \sqrt{[(I+1)^2-\nu^2]}\,a_+(I, Y)\,\delta(II \\
&\quad \sqrt{[I^2 \quad \nu^2]}\,a_-(I, Y)\,\delta(II' \quad 1) + b(I, Y)\,\delta(II')\big]
\end{aligned} \tag{3.10}
$$

, $a_0$ and $b$ being unknown functions of $I$ and $Y$.

([16]) R. E. MARSHAK, S. OKUBO and E. C. G. SUDARSHAN: *Phys. Rev.*, 106, 599 (1957).

TABLE III. – *Consequences of charge-independence for electromagnetic form factors.*

IR (3, 0)

$$\begin{aligned}
F(\tfrac{3}{2}\ -\tfrac{3}{2}\ \ 1)-F(\tfrac{3}{2}\ -\tfrac{1}{2}\ \ 1) &= F(\tfrac{3}{2}\ -\tfrac{1}{2}\ \ 1)-F(\tfrac{3}{2}\ \ \tfrac{1}{2}\ \ 1)=\\
&= F(\tfrac{3}{2}\ \ \tfrac{1}{2}\ \ 1)-F(\tfrac{3}{2}\ \ \tfrac{3}{2}\ \ 1)\\
F(1\ -1\ \ 0)-F(1\ \ 0\ \ 0) &= F(1\ \ 0\ \ 0)-F(1\ \ 1\ \ 0)
\end{aligned}$$

IR (2, 2)

$$\begin{aligned}
F(1\ -1\ \ Y)-F(1\ \ 0\ \ Y) &= F(1\ \ 0\ \ Y)-F(1\ \ 1\ \ Y);\\
&\quad Y=2,0 \text{ and } -2\\
F(\tfrac{3}{2}\ -\tfrac{3}{2}\ \ Y)-F(\tfrac{3}{2}\ -\tfrac{1}{2}\ \ Y) &= F(\tfrac{3}{2}\ -\tfrac{1}{2}\ \ Y)-F(\tfrac{3}{2}\ \ \tfrac{1}{2}\ \ Y)=\\
&= F(\tfrac{3}{2}\ \ \tfrac{1}{2}\ \ Y)-F(\tfrac{3}{2}\ \ \tfrac{3}{2}\ \ Y);\\
&\quad Y=1 \text{ and } -1\\
F(2\ -2\ \ 0)-F(2\ -1\ \ 0) &= F(2\ -1\ \ 0)-F(2\ \ 0\ \ 0)=\\
&= F(2\ \ 0\ \ 0)-F(2\ \ 1\ \ 0)=\\
&= F(2\ \ 1\ \ 0)-F(2\ \ 2\ \ 0)\\
F_T(\tfrac{3}{2}\ -\tfrac{1}{2}\ \ Y;\ \tfrac{1}{2}\ -\tfrac{1}{2}\ \ Y) &= F_T(\tfrac{3}{2}\ \ \tfrac{1}{2}\ \ Y;\ \tfrac{1}{2}\ \ \tfrac{1}{2}\ \ Y);\quad Y=1 \text{ and } -1\\
F_T(0\ \ 0\ \ 0;\ 2\ \ 0\ \ 0) &= 0\\
2F_T(2\ -1\ \ 0;\ 1\ -1\ \ 0) &= \sqrt{3}\,F_T(2\ \ 0\ \ 0;\ 1\ \ 0\ \ 0)=\\
&= 2F_T(2\ \ 1\ \ 0;\ 1\ \ 1\ \ 0)\,.
\end{aligned}$$

IR (4, 1)

$$\begin{aligned}
F(\tfrac{5}{2}\ -\tfrac{5}{2}\ \ 1)-F(\tfrac{5}{2}\ -\tfrac{3}{2}\ \ 1) &= F(\tfrac{5}{2}\ -\tfrac{3}{2}\ \ 1)-F(\tfrac{5}{2}\ -\tfrac{1}{2}\ \ 1)=\\
&= F(\tfrac{5}{2}\ -\tfrac{1}{2}\ \ 1)-F(\tfrac{5}{2}\ \ \tfrac{1}{2}\ \ 1)=\\
&= F(\tfrac{5}{2}\ \ \tfrac{1}{2}\ \ 1)-F(\tfrac{5}{2}\ \ \tfrac{3}{2}\ \ 1)=\\
&= F(\tfrac{5}{2}\ \ \tfrac{3}{2}\ \ 1)-F(\tfrac{5}{2}\ \ \tfrac{5}{2}\ \ 1)\\
F(2\ -2\ \ Y)-F(2\ -1\ \ Y) &= F(2\ -1\ \ Y)-F(2\ \ 0\ \ Y)=\\
&= F(2\ \ 0\ \ Y)-F(2\ \ 1\ \ Y)=\\
&= F(2\ \ 1\ \ Y)-F(2\ \ 2\ \ Y);\\
&\quad Y=2 \text{ and } 0\\
F(\tfrac{3}{2}\ -\tfrac{3}{2}\ \ Y)-F(\tfrac{3}{2}\ -\tfrac{1}{2}\ \ Y) &= F(\tfrac{3}{2}\ -\tfrac{1}{2}\ \ Y)-F(\tfrac{3}{2}\ \ \tfrac{1}{2}\ \ Y)=\\
&= F(\tfrac{3}{2}\ \ \tfrac{1}{2}\ \ Y)-F(\tfrac{3}{2}\ \ \tfrac{3}{2}\ \ Y);\\
&\quad Y=1 \text{ and } -1\\
F(1\ -1\ \ Y)-F(1\ \ 0\ \ Y) &= F(1\ \ 0\ \ Y)-F(1\ \ 1\ \ Y);\\
&\quad Y=0 \text{ and } -2\\
\sqrt{3}\,F_T(\tfrac{5}{2}\ -\tfrac{3}{2}\ \ 1;\ \tfrac{3}{2}\ -\tfrac{3}{2}\ \ 1) &= \sqrt{2}\,F_T(\tfrac{5}{2}\ -\tfrac{1}{2}\ \ 1;\ \tfrac{3}{2}\ -\tfrac{1}{2}\ \ 1)=\\
&= \sqrt{2}\,F_T(\tfrac{5}{2}\ \ \tfrac{1}{2}\ \ 1;\ \tfrac{3}{2}\ \ \tfrac{1}{2}\ \ 1)=\\
&= \sqrt{3}\,F_T(\tfrac{5}{2}\ \ \tfrac{3}{2}\ \ 1;\ \tfrac{3}{2}\ \ \tfrac{3}{2}\ \ 1)\\
F_T(2\ -1\ \ 0;\ 1\ -1\ \ 0) &= \sqrt{3}\,F_T(2\ \ 0\ \ 0;\ 1\ \ 0\ \ 0)=\\
&= F_T(2\ \ 1\ \ 0;\ 1\ \ 1\ \ 0)\\
-\tfrac{1}{2}\ -1;\ \tfrac{1}{2}\ -\tfrac{1}{2}\ -1) &= F_T(\tfrac{3}{2}\ \ \tfrac{1}{2}\ -1;\ \tfrac{1}{2}\ \ \tfrac{1}{2}\ -1)\,.
\end{aligned}$$

## 4. – Electromagnetic relations of the second kind: self-energies and mass formulae.

We now consider electromagnetic relations of the second kind for quantities with a quadratic dependence on the electric charge-current density with application to self-energies especially in mind. Calculations proceed along the same lines as in Sect. **3** except that we here use the specialization $\varphi(I, Y) \to$ $\to E(I, Y)$ of (2.9) where $E(I, Y)$ is given (Okubo's [7] second-order formula) by

$$E(I, Y) = a + bY + cZ(I, Y) + dY^2 + eYZ(I, Y) + fZ^2(I, Y)\,, \tag{4.1}$$

where $Z(I, Y) = I(I+1) - Y^2/4$. Here we need not use a tracelessness condition to eliminate the constant term $a$. Our chief interest is in self-energies and the constant term of the electromagnetic self-energy can be absorbed into the self-energy due to $SU_3$-invariant interactions which is constant for all members of a unitary multiplet.

We cannot illustrate the method of the present Section using the stable baryon octet because eq. (4.1) fails to give any information. However, the relations of the first kind that hold for the electromagnetic self-energies of the baryons and the mass formulae that follow from them are of sufficient interest to merit inclusion. If we assume, first of all, that the observed baryon mass (denoted by capital $M$) is the sum of a self-energy contribution due to strong interactions, not necessarily exactly invariant with respect to $SU_3$, and an electromagnetic self-energy contribution, we may use the relationships [17]

$$E(\Sigma^+) \quad E(\mathrm{p}) \qquad E(\Sigma^-) = E(\Xi^- \qquad E(\mathrm{n}) = E(\Xi^0)$$

for baryon electromagnetic self-energies to deduce the mass formula [8]

$$M(\mathrm{p}) \quad M(\mathrm{n}) + M(\Xi^0) \quad M(\Xi^- + M(\Sigma^- - M(\Sigma^+) = 0$$

Similarly if we include the transition self-energy $M_T(\Sigma^0, \Lambda) = E_T(\Sigma^0, \Lambda)$, we may use $\sqrt{3}\,E_T(\Sigma^0, \Lambda) = E(\mathrm{n}) - E(\Sigma^0) = E(\Xi^0) - E(\Sigma^0)$, to derive the results

$$\begin{aligned} \sqrt{3}\,M_T(\Sigma^0, \Lambda) &= M(\mathrm{n}) - M(\mathrm{p}) \quad M(\Sigma^0) + M(\Sigma^+) \\ &= M(\Xi^0) \quad M(\Xi \qquad M(\Sigma^0) + M(\Sigma^- \end{aligned} \tag{4.3}$$

which are not independent by virtue of (4.2). It is to be noted that, in addition to depending on $M_T(\Sigma^0, \Lambda)$, these results only depend on mass differences within isotopic multiplets. Accordingly it is to be hoped that they are useful in all orders of perturbation of the $SU_3$-invariant strong interactions by merely charge-independent interactions. If we explicitly make the assumption that the perturbation involved is a first-order effect, we can use the relationship [2]

$$m(\Sigma) + 3\,m(\Lambda) \quad 2\,m(\Xi) + 2\,m(\mathcal{N} \tag{4.5}$$

---

[17] See remarks that follows eq. (2.11)

TABLE IV. – *Electromagnetic self-energies.*

IR (3, 0)

$$E\left(\tfrac{3}{2}\;-\tfrac{3}{2}\;\;1\right)=E(1\;-1\;\;0)=E\left(\tfrac{1}{2}\;-\tfrac{1}{2}\;-1\right)=E(0\;\;0\;-2)=a+b+c$$
$$E\left(\tfrac{3}{2}\;-\tfrac{1}{2}\;\;1\right)=E(1\;\;0\;\;0)=E\left(\tfrac{1}{2}\;\;\tfrac{1}{2}\;-1\right)=a$$
$$E\left(\tfrac{3}{2}\;\;\tfrac{1}{2}\;\;1\right)=E(1\;\;1\;\;0)=a-b+c$$
$$E\left(\tfrac{3}{2}\;\;\tfrac{3}{2}\;\;1\right)=a-2b+4c$$

IR (2, 2)

$$E(2\;-2\;\;0)=E\left(\tfrac{3}{2}\;-\tfrac{3}{2}\;-1\right)=E(1\;-1\;-2)=a+2b-6c+4d-12e+12f$$
$$E\left(\tfrac{3}{2}\;-\tfrac{3}{2}\;-1\right)=E(1\;\;0\;-2)=a+b-21c+d-21e+147f$$
$$E(1\;-1\;\;2)=E(1\;\;1\;-2)=a-36c+432f$$
$$E(1\;\;0\;\;2)=E\left(\tfrac{3}{2}\;\;\tfrac{3}{2}\;-1\right)=a-b-21c+d+21e+147f$$
$$E(1\;\;1\;\;2)=E\left(\tfrac{3}{2}\;\;\tfrac{3}{2}\;\;1\right)=E(2\;\;2\;\;0)=a-2b-6c+4d+12e+12f$$
$$E(2\;-1\;\;0)=a+b-6c+d-6e+27f$$
$$E(1\;-1\;\;0)=a+b-18c+d-18e+123f$$
$$E_T(2\;-1\;\;0;\;1\;-1\;\;0)=3\sqrt{5}(-c-e+8f)$$
$$E\left(\tfrac{3}{2}\;-\tfrac{1}{2}\;-1\right)=a+b-11c+d-11e+201f$$
$$E\left(\tfrac{1}{2}\;-\tfrac{1}{2}\;-1\right)=a+b-13c+d-13e+249f$$
$$E_T\left(\tfrac{3}{2}\;-\tfrac{1}{2}\;-1;\;\tfrac{1}{2}\;-\tfrac{1}{2}\;-1\right)=4\sqrt{5}(-c-e+8f)$$
$$E\left(\tfrac{3}{2}\;\;\tfrac{1}{2}\;-1\right)=a-16c+112f$$
$$E\left(\tfrac{1}{2}\;\;\tfrac{1}{2}\;-1\right)=a-32c+368f$$
$$E_T\left(\tfrac{3}{2}\;\;\tfrac{1}{2}\;-1;\;\tfrac{1}{2}\;\;\tfrac{1}{2}\;-1\right)=4\sqrt{5}(-c+16f)$$
$$E\left(\tfrac{3}{2}\;\;\tfrac{1}{2}\;\;1\right)=a-b-11c+d+11e+67f$$
$$E\left(\tfrac{1}{2}\;\;\tfrac{1}{2}\;\;1\right)=a-b-13c+d+13e+83f$$
$$E_T\left(\tfrac{3}{2}\;\;\tfrac{1}{2}\;\;1;\;\tfrac{1}{2}\;\;\tfrac{1}{2}\;\;1\right)=4\sqrt{5}(-c+e+8f)$$
$$E(2\;\;1\;\;0)=a-b-6c+d+6e+27f$$
$$E(1\;\;1\;\;0)=a-b-18c+d+18e+123f$$
$$E_T(2\;\;1\;\;0;\;1\;\;1\;\;0)=3\sqrt{5}(-c+e+8f)$$
$$E\left(\tfrac{3}{2}\;-\tfrac{1}{2}\;\;1\right)=a-16c+112f$$
$$E\left(\tfrac{1}{2}\;-\tfrac{1}{2}\;\;1\right)=a-32c+368f$$
$$E_T\left(\tfrac{3}{2}\;-\tfrac{1}{2}\;\;1;\;\tfrac{1}{2}\;-\tfrac{1}{2}\;\;1\right)=4\sqrt{5}(-c+16f)$$
$$E(2\;\;0\;\;0)=a-6c+192f$$
$$E(1\;\;0\;\;0)=a-18c+192f$$
$$E(0\;\;0\;\;0)=a-24c+256f$$
$$E_T(2\;\;0\;\;0;\;1\;\;0\;\;0)=\sqrt{15}(-2c+16f)$$
$$E_T(1\;\;0\;\;0;\;0\;\;0\;\;0)=\sqrt{3}(-8c+112f)$$
$$E_T(2\;\;0\;\;0;\;0\;\;0\;\;0)=16\sqrt{5}f\,.$$

IR (4, 1)

$$E\left(\tfrac{5}{2}\;-\tfrac{5}{2}\;\;1\right)=E(2\;-2\;\;0)=E\left(\tfrac{3}{2}\;-\tfrac{3}{2}\;-1\right)=E(1\;-1\;-2)=E\left(\tfrac{1}{2}\;-\tfrac{1}{2}\;-3\right)=a+2b-50c+4d-100e+500f$$
$$E(2\;-2\;\;2)=E\left(\tfrac{1}{2}\;\;\tfrac{1}{2}\;-3\right)=a+b-85c+d-85e+1445f$$
$$E(2\;-1\;\;2)=E(1\;\;1\;-2)=a-60c+720f$$
$$E(2\;\;0\;\;2)=E\left(\tfrac{3}{2}\;\;\tfrac{3}{2}\;-1\right)=a-b-35c+d+35e+245f$$

TABLE IV (*continued*).

$$
\begin{aligned}
&E\,(2 \quad 1 \quad 2) = E(2 \quad 2 \quad 0) = a - 2b - 10c + 4d + 20e + 20f\\
&E\,(2 \quad 2 \quad 2) = E(\tfrac{5}{2} \quad \tfrac{5}{2} \quad 1) = a - 3b + 15c + 9d - 45e + 45f\\
&E\,(2 \; -1 \quad 0) = a + b - 40c + d - 40e + 316f\\
&E\,(1 \; -1 \quad 0) = a + b - 80c + d - 80e + 1325f\\
&E_T(2 \; -1 \quad 0;\; 1 \; -1 \quad 0) = 15(-c - e + 24f)\\
&E\,(\tfrac{5}{2} \; -\tfrac{3}{2} \quad 1) = a + b - 37c + d - 37e + 293f\\
&E\,(\tfrac{3}{2} \; -\tfrac{3}{2} \quad 1) = a + b - 83c + d - 83e + 1397f\\
&E_T(\tfrac{5}{2} \; -\tfrac{3}{2} \quad 1;\; \tfrac{3}{2} \; -\tfrac{3}{2} \quad 1) = 4\sqrt{6}(-c - e + 24f)\\
&E\,(\tfrac{5}{2} \; -\tfrac{1}{2} \quad 1) = a - 24c + 144f\\
&E\,(\tfrac{3}{2} \; -\tfrac{1}{2} \quad 1) = a - 56c + 656f\\
&E_T(\tfrac{5}{2} \; -\tfrac{1}{2} \quad 1;\; \tfrac{3}{2} \; -\tfrac{1}{2} \quad 1) = 12(-c + 16f)\\
&E\,(2 \quad 0 \quad 0) = a - 15c + 240f\\
&E\,(1 \quad 0 \quad 0) = a - 50c + 560f\\
&E_T(2 \quad 0 \quad 0;\; 1 \quad 0 \quad 0) = 10\sqrt{3}(-c + 16f)\\
&E\,(\tfrac{5}{2} \quad \tfrac{1}{2} \quad 1) = a - b - 11c + d + 11e + 53f\\
&E\,(\tfrac{3}{2} \quad \tfrac{1}{2} \quad 1) = a - b - 29c + d + 29e + 197f\\
&E_T(\tfrac{5}{2} \quad \tfrac{1}{2} \quad 1;\; \tfrac{3}{2} \quad \tfrac{1}{2} \quad 1) = 12(-c + e + 8f)\\
&E\,(2 \quad 1 \quad 0) = a - b - 20c + d + 20e + 125f\\
&E\,(1 \quad 1 \quad 0) = a - b - 20c + d + 20e + 125f\\
&E_T(2 \quad 1 \quad 0;\; 1 \quad 1 \quad 0) = 15(-c + e + 8f)\\
&E\,(\tfrac{5}{2} \quad \tfrac{3}{2} \quad 1) = a - 2b + 2c + 4d - 4e + 20f\\
&E\,(\tfrac{3}{2} \quad \tfrac{3}{2} \quad 1) = a - 2b - 2c + 4d + 4e + 20f\\
&E_T(\tfrac{5}{2} \quad \tfrac{3}{2} \quad 1;\; \tfrac{3}{2} \quad \tfrac{3}{2} \quad 1) = 4\sqrt{6}(-c + 2e)\\
&E\,(1 \quad 0 \; -2) = a + b - 55c + d - 55e + 725f\\
&E\,(0 \quad 0 \; -2) = a + b - 65c + d - 65e + 965f\\
&E_T(1 \quad 0 \; -2;\; 0 \quad 0 \; -2) = 10\sqrt{6}(-c - e + 24f)\\
&E\,(\tfrac{3}{2} \quad \tfrac{1}{2} \; -1) = a - 40c + 400f\\
&E\,(\tfrac{1}{2} \quad \tfrac{1}{2} \; -1) = a - 40c + 400f\\
&E_T(\tfrac{3}{2} \quad \tfrac{1}{2} \; -1;\; \tfrac{1}{2} \quad \tfrac{1}{2} \; -1) = 20(-c + 16f)\\
&E\,(\tfrac{3}{2} \; -\tfrac{1}{2} \; -1) = a + b - 45c + d - 45e + 485f\\
&E\,(\tfrac{1}{2} \; -\tfrac{1}{2} \; -1) = a + b - 75c + d - 75e + 1205f\\
&E_T(\tfrac{3}{2} \; -\tfrac{1}{2} \; -1;\; \tfrac{1}{2} \; -\tfrac{1}{2} \; -1) = 20(-c - e + 24f)\,.
\end{aligned}
$$

between the self-energies of components of baryon isotopic multiplets due to strong interactions, to predict that the result

$$2\sqrt{3}\, M_T(\Sigma^0, \Lambda) \qquad 2\, M(\mathrm{n}) + 2\, M(\Xi^0) \qquad 3\, M(\Lambda) \qquad M(\Sigma^0)\,,$$

is valid. In the above discussion of mass formulae we have not taken into account the fact that the results used for electromagnetic self-energies depend on the neglect of strong interactions that break $SU_3$-invariance. In fact [7], eqs. (4.2), (4.4) are strictly accurate only when the strong interactions that break $SU_3$-invariance are absent, and eq. (4.6) holds only when they are no more than a first-order perturbation of the $SU_3$-invariant strong interactions. A

TABLE V. – *Consequences of charge-independence for electromagnetic self-energies.*

IR (3, 0)

$$E(\tfrac{3}{2}\;-\tfrac{3}{2}\;\;1)-E(\tfrac{3}{2}\;\;\tfrac{3}{2}\;\;1)=3[E(\tfrac{3}{2}\;-\tfrac{1}{2}\;\;1)-E(\tfrac{3}{2}\;\;\tfrac{1}{2}\;\;1)]$$

IR (2, 2)

$$E(\tfrac{3}{2}\;-\tfrac{3}{2}\;\;Y)-E(\tfrac{3}{2}\;\;\tfrac{3}{2}\;\;Y)=3[E(\tfrac{3}{2}\;-\tfrac{1}{2}\;\;Y)-E(\tfrac{3}{2}\;\;\tfrac{1}{2}\;\;1)],\quad \text{for } Y=1 \text{ and } -1$$

$$E(2\;-2\;\;0)-E(2\;\;2\;\;0)=2[E(2\;-1\;\;0)-E(2\;\;1\;\;0)]$$

$$E(2\;-2\;\;0)+6E(2\;\;0\;\;0)+E(2\;\;2\;\;0)==4[E(2\;-1)+E(2\;\;1\;\;0)]$$

$$E_\gamma(2\;-1\;\;0;\,1\;-1\;\;0)+E_\gamma(2\;\;1\;\;0;\,1\;\;0\;\;0)==\sqrt{3}\,E_\gamma(2\;\;0\;\;0;\,1\;\;0\;\;0)$$

IR (4, 1)

$$E(\tfrac{3}{2}\;-\tfrac{3}{2}\;\;Y)-E(\tfrac{3}{2}\;\;\tfrac{3}{2}\;\;Y)=3[E(\tfrac{3}{2}\;-\tfrac{1}{2}\;\;Y)-E(\tfrac{3}{2}\;\;\tfrac{1}{2}\;\;Y)],\quad \text{for } Y=1 \text{ and } -1$$

$$E(2\;-2\;\;Y)-E(2\;\;2\;\;Y)=2[E(2\;-1\;\;Y)-E(2\;\;1\;\;Y)],\quad \text{for } Y=2 \text{ and } 0$$

$$E(2\;-2\;\;Y)+6E(2\;\;0\;\;Y)+E(2\;\;2\;\;Y)==4[E(2\;-1\;\;Y)+E(2\;\;1\;\;Y)],\quad \text{for } Y=2 \text{ and } 0$$

$$3[E(\tfrac{5}{2}\;-\tfrac{5}{2}\;\;1)-E(\tfrac{5}{2}\;\;\tfrac{5}{2}\;\;1)]=5[E(\tfrac{5}{2}\;-\tfrac{3}{2}\;\;1)-E(\tfrac{5}{2}\;\;\tfrac{3}{2}\;\;1)]==15[E(\tfrac{5}{2}\;-\tfrac{1}{2}\;\;1)-E(\tfrac{5}{2}\;\;\tfrac{1}{2}\;\;1)]$$

$$E(\tfrac{5}{2}\;-\tfrac{5}{2}\;\;1)+2E(\tfrac{5}{2}\;-\tfrac{1}{2}\;\;1)+2E(\tfrac{5}{2}\;\;\tfrac{1}{2}\;\;1)+E(\tfrac{5}{2}\;\;\tfrac{5}{2}\;\;1)==3[E(\tfrac{5}{2}\;-\tfrac{3}{2}\;\;1)+E(\tfrac{5}{2}\;\;\tfrac{3}{2}\;\;1)]$$

$$E_\gamma(\tfrac{5}{2}\;-\tfrac{3}{2}\;\;1;\,\tfrac{3}{2}\;-\tfrac{3}{2}\;\;1)-E_\gamma(\tfrac{5}{2}\;\;\tfrac{3}{2}\;\;1;\,\tfrac{3}{2}\;\;\tfrac{3}{2}\;\;1)==\sqrt{6}[E_\gamma(\tfrac{5}{2}\;-\tfrac{1}{2}\;\;1;\,\tfrac{3}{2}\;-\tfrac{1}{2}\;\;1)-E_\gamma(\tfrac{5}{2}\;\;\tfrac{1}{2}\;\;1;\,\tfrac{3}{2}\;\;\tfrac{1}{2}\;\;1)]$$

$$E_\gamma(\tfrac{5}{2}\;-\tfrac{3}{2}\;\;1;\,\tfrac{3}{2}\;-\tfrac{3}{2}\;\;1)+E_\gamma(\tfrac{5}{2}\;\;\tfrac{3}{2}\;\;1;\,\tfrac{3}{2}\;\;\tfrac{3}{2}\;\;1)==\sqrt{\tfrac{2}{3}}[E_\gamma(\tfrac{5}{2}\;-\tfrac{1}{2}\;\;1;\,\tfrac{3}{2}\;-\tfrac{1}{2}\;\;1)-E_\gamma(\tfrac{5}{2}\;\;\tfrac{3}{2}\;\;1;\,\tfrac{3}{2}\;\;\tfrac{1}{2}\;\;1)]$$

$$E_\gamma(2\;-1\;\;0;\,1\;-1\;\;0)+E_\gamma(2\;\;1\;\;0;\,1\;\;1\;\;0)==\sqrt{3}\,E_\gamma(2\;\;0\;\;0;\,1\;\;0\;\;0)\,.$$

comment about the transition mass seems necessary. Unlike the transition form factor, which is directly accessible from the life time of the radiative decay $\Sigma^0\to\Lambda+\gamma$, the transition mass enters only into the definition of exact eigenstates of mass and a measure of it is only indirectly obtained from a measure of the isotopic impurity of $\Lambda$.

In the case of the decuplet there are no transition masses. Proceeding via similar assumptions to those made for the baryons, we find four relations

$$
\begin{aligned}
&M(\mathcal{N}^{*++})\quad M(\mathcal{N}^{*-})\quad 3[M(\mathcal{N}^{*+})-M(\mathcal{N}^{*0})]\,,\\
(4.7)\qquad &M(\mathcal{N}^{*0})\quad M(\mathcal{N}^{*-})\quad M(Y^{*0})-M(Y^{*-})=M(\Xi^{*0})\quad M(\Xi^{*-})\\
&M(\mathcal{N}^{*+})\quad M(\mathcal{N}^{*0})\quad M(Y^{*+})-M(Y^{*0})\,,
\end{aligned}
$$

involving only mass differences within isotopic multiplets. The first relation here is a consequence simply of the Wigner-Eckart theorem for the isospin subgroup of $SU_3$, and hence is certainly independent of the nature of the breaking of exact $SU_3$ invariance of strong interactions. Supposing, for the strong interaction (but not for the electromagnetic) self-energy contributions, we take account of the breaking of the invariance of strong interactions with respect to $SU_3$ to first order, then we have

$$
\begin{aligned}
M(\Xi^{*-}) &= 2\,M(Y^{*-}) - M(\mathcal{N}^{*-}) \\
M(\Omega^-) &= 3\,M(\Xi^{*-}) - 3\,M(Y^{*-}) + M(\mathcal{N}^{*-})\,,
\end{aligned}
\tag{4.8}
$$

of which only the second survives in the second order.

Parametric expression for electromagnetic self-energies for other unitary multiplets occupy Table IV, whilst relations, like the first one in (4.7), which hold even when strong interactions are no more than charge-independent are displayed in Table V.

## 5. – Discussion.

We have in previous sections confined attention to electromagnetic matrix elements between components of the same unitary multiplet. However, no essential generalization of the work of Sect. **2** is required in the derivation of relations of the first kind for the radiative decays of one unitary multiplet into an equivalent but not identical multiplet plus an arbitrary number of photons. Additional relations (of the second kind) valid for one-photon decays follow as in Sect. **3**. An example of physical interest is afforded by the radiative decay of vector mesons into pseudoscalar mesons. When we recall that charge conjugation invariance applies to the situation, we readily reproduce certain results [18] due to OKUBO.

It might appear, at first sight, that generalization of the given method to allow treatment, for example, of radiative decays of baryon resonances associated [19] with the decuplet IR (3, 0) of $SU_3$ into baryons is less immediate. However, the method of section two applies without modification: $\Phi(-\mathscr{Y})$ is still diagonal with respect to $I$, $\nu$ and $Y$ and independent of $\nu$ even where its matrix elements between states of different IR's of $SU_3$ are concerned. It turns out that the nine allowed radiative decay amplitudes can be expressed

[18] Equation (10) of ref. [7].
[19] S. L. GLASHOW and J. J. SAKURAI: *Nuovo Cimento*, **26**, 622 (1962).

in terms of two unknown quantities. The seven relations of the first kind that hereby arise are displayed in Table VI, together with one additional result (of the second kind) that holds for one-photon decay amplitudes.

TABLE VI. – *Amplitudes for radiative decay of a decuplet into an octet.*

$$\begin{array}{llllllll}
\langle(1,1)\,\tfrac{1}{2} & -\tfrac{1}{2} & 1|D|(3,0)\,\tfrac{3}{2} & -\tfrac{1}{2} & 1\rangle = D(1,0) \\
\langle(1,1)\,\tfrac{1}{2} & \tfrac{1}{2} & 1|D|(3,0)\,\tfrac{3}{2} & \tfrac{1}{2} & 1\rangle = D(\tfrac{1}{2},-1) \\
\langle(1,1)\,1 & -1 & 0|D|(3,0)\,1 & -1 & 0\rangle = 0 \\
\langle(1,1)\,1 & 0 & 0|D|(3,0)\,1 & 0 & 0\rangle = \tfrac{1}{2}D(1,0) \\
\langle(1,1)\,1 & 1 & 0|D|(3,0)\,1 & 1 & 0\rangle = D(\tfrac{1}{2},-1) \\
\langle(1,1)\,\tfrac{1}{2} & -\tfrac{1}{2} & -1|D|(3,0)\,\tfrac{1}{2} & -\tfrac{1}{2} & -1\rangle = 0 \\
\langle(1,1)\,\tfrac{1}{2} & \tfrac{1}{2} & -1|D|(3,0)\,\tfrac{1}{2} & \tfrac{1}{2} & -1\rangle = -D(1,0) \\
\langle(1,1)\,0 & 0 & 0|D|(3,0)\,1 & 0 & 0\rangle = -(\sqrt{3}/2)\,D(1,0)\,.
\end{array}$$

For one-photon decays, we have relations

$$\begin{aligned}
M(\mathcal{N}^{+}_{*}\to \mathrm{p}+\gamma) &= M(\mathrm{Y}^{+}_{1*}\to\Sigma^{+}+\gamma) \\
M(\mathrm{Y}^{-}_{1*}\to\Sigma^{-}+\gamma) &= M(\Xi^{-}_{*}\to\Xi^{-}+\gamma) = 0 \\
M(\mathcal{N}^{0}_{*}\to \mathrm{n}+\gamma) &= 2M(\mathrm{Y}^{0}_{1*}\to\Sigma^{0}+\gamma) = \\
&= -M(\Xi^{0}_{*}\to\Xi^{0}+\gamma) = -(2/\sqrt{3})\,M(\mathrm{Y}^{0}_{1*}\to\Lambda+\gamma),
\end{aligned}$$

of the first kind, and

$$M(\mathcal{N}^{+}_{*}\to \mathrm{p}+\gamma) = M(\mathcal{N}^{0}_{*}\to \mathrm{n}+\mathrm{p})$$

of the second kind.

We have just seen that our method can handle, as far as relations of the first kind are concerned, any such « nondiagonal » radiative decay amplitude. However, to obtain results of the second kind for one- or two-photon amplitudes, we need generalizations of the Okubo formulae to all nonvanishing matrix elements of $\mathcal{Y}$ between states of different IR's of $SU_3$. The required first-order formulae have indeed been developed ([20]) and will be discussed along with applications elsewhere.

In our discussion of electromagnetic properties we have consistently disregarded the strong interactions which break exact $SU_3$-invariance and produce mass splitting within unitary multiplets. If these are taken into account even to first order the numbers of unknown parameters that appear in our calculations already suffer a considerable increase. For example the number of parameters involved ([7]) in a calculation of baryon magnetic moments is

([20]) D. LURIÉ and A. J. MACFARLANE: to be published.

now eight in contrast to the two needed in the absence of the mass splitting interactions. The possibility of practically applicable relations emerging at this level of generality is remote; it appears that the situation ought to be studied in terms of more detailed dynamical models.

## RIASSUNTO (*)

Dimostriamo che si può ottenere formalmente una teoria in cui le interazioni forti invarianti rispetto ad $SU_3$ sono perturbate solo da interazioni elettromagnetiche con una trasformazione unitaria di una teoria in cui le interazioni forti invarianti rispetto ad $SU_3$ sono perturbate da interazioni indipendenti soltanto dalla carica. Sfruttando informazioni facilmente ottenibili su quest'ultima teoria, possiamo dare una rapida deduzione delle relazioni fra le proprietà elettromagnetiche di varie particelle e risonanze. Mentre si riferiscono molti nuovi risultati, si mette in particolare rilievo l'efficacia del metodo ed il riconoscimento della natura esatta delle ipotesi necessarie per dedurre i risultati. Per esempio, delle sette familiari relazioni fra i momenti magnetici dei barioni solo due richiedono l'uso dell'ipotesi che ci sia una dipendenza lineare fra i momenti magnetici e la densità della corrente di carica elettromagnetica; le altre sono esempi di relazioni valide in forma identica per tutte le proprietà elettromagnetiche dei barioni — fattori di forma elettrici e magnetici, autoenergie elettromagnetiche, ampiezze dello scattering di Compton, etc.

(*) *Traduzione a cura della Redazione.*