Reprinted from
JOURNAL OF MATHEMATICAL PHYSICS VOLUME 6, NUMBER 8 AUGUST 1965
Printed in U.S.A.

Concerning Space–Time, Symmetry Groups, and Charge Conservation*

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(Received 26 October 1964)

Under the restriction to a symmetry group whose generators can be linearly expressed in terms of the generators of the Lorentz group and of an internal symmetry group, we show that charge conservation implies that the symmetry group is a direct product of the Lorentz group and the internal symmetry group. Only local structure of the groups and their unitary representations are considered.

THE problem of combining relativistic invariance and internal symmetries has recently been discussed by several authors.[1] Since the multiplet masses are only approximately the same, the only way of reconciling exact invariance under both the internal symmetry group and the relativity group is to require that the elements of the two groups do not commute in general. Let us assume[2] that the full invariance group of the physical (strongly interacting) system is a Lie group (which has as subgroups the internal symmetry group, assumed to be simple, and the inhomogeneous Lorentz group) which has a Lie algebra A whose elements can be expressed as a linear combination of the elements of the Lie algebras of the internal and Lorentz groups. It can then be shown[3] that, if a complete set of commuting generators of the semisimple internal symmetry algebra S commute with the generators of the inhomogeneous Lorentz algebra L, then the algebra A is a direct sum of the algebras S and L.

In this paper we wish to extend this result to the case when only a single generator of S commutes with all elements of L. This framework is of particular interest since electric charge is conserved in all known interactions and it is a generator, or associated with a generator of the internal symmetry group which we expect to be relativistically invariant. We show that as long as we are interested in unitary representations only a direct sum algebra results; this is disappointing since there is then no possible explanation of the mass splittings within a multiplet compatible with exact invariance under the group.

Theorem. Let Q be any generator of S. If Q commutes with every element of L (and if the set of elements of S and L are closed under commutation), then every element L_A of L can be decomposed in the form with $L_A = L_A^0 + L_A^1$, L_A^1 being a linear combination of elements of S and L_A^0 commuting with every element of S. Further, both L_A^0 and L_A^1 satisfy the same commutation relations as the elements L_A of the Lorentz algebra.

Proof. Since the Lie algebra S is simple, there exists a Cartan–Weyl basis H_l, E_α such that

$$[H_l,\ E_\alpha] = r_l(\alpha)E_\alpha, \tag{2a}$$

$$[H_l,\ H_m] = 0, \tag{2b}$$

$$[E_\alpha,\ E_\beta] = N_{\alpha\beta}E_{\alpha+\beta}, \qquad r(\alpha) + r(\beta) \not\equiv 0, \tag{2c}$$

$$[E_\alpha,\ E_{-\alpha}] = \sum_l r_l(\alpha)H_l, \tag{2d}$$

No generality is lost by taking H_1 to be a multiple of Q so that

$$[H_1,\ L_A] = 0. \tag{3}$$

By hypothesis we can write

$$[E_\alpha,\ L_A] = \sum_\beta a(\alpha A\beta)E_\beta + \sum_l a(\alpha Al)H_l + \sum_B a(\alpha AB)L_B.$$

Hence, if $r_1(\alpha) \neq 0$ we deduce

$$[E_\alpha,\ L_A] = \sum_\beta a(\alpha A\beta)\delta\{r_1(\alpha) - r_1(\beta)\}E_\beta.$$

For those cases where $r_1(\alpha) = 0$, we can use (2c) to deduce the general expression

$$[E_\alpha,\ L_A] = \sum_\beta a(\alpha A\beta)\delta\{r_1(\alpha) - r_1(\beta)\}E_\beta + \sum_m a(\alpha Am)\delta\{r_1(\alpha)\}H_m. \tag{4}$$

Similarly from (2b) and (3) we deduce

$$[H_l,\ L_A] = \sum_\beta b(lA\beta)\delta\{r_1(\beta)\}E_\beta + \sum_m b(lAm)H_m. \tag{5}$$

* Supported in part by the United States Atomic Energy Commission.

[1] F. Lurçat and L. Michel, Nuovo Cimento **21**, 574 (1961); *Proceedings of the Coral Gables Conference on Symmetry Principles at High Energies*, edited by B. Kursunoglu and A. Perlmutter (W. J. Freeman Company, San Francisco, 1964); A. O. Barut, Nuovo Cimento **32**, 234 (1964); B. Kursunoglu, Phys. Rev. **135**, B761 (1964).

[2] Following W. D. McGlinn, Phys. Rev. Letters **12**, 469 (1964).

[3] F. Coester, M. Hamermesh, and W. D. McGlinn; Phys. Rev. **135**, B451 (1964); M. E. Mayer, H. J. Schnitzer, E. C. G. Sudarshan, R. Acharya, and M. Y. Han, Phys. Rev. **136**, B888 (1964). A Beskow and V. Ottoson, Nuovo Cimento **34**, 248 (1964).

Evaluating the double commutator

$$[H_l, [E_\alpha, \quad L_A]] = \sum_\beta a(\alpha A\beta) r_l(\beta) \delta\{r_1(\alpha) - r_1(\beta)\} E_\beta$$

using the Jacobi identity, we get the expression

$$\begin{aligned}[][[H_l, \quad E_\alpha], L_A] - [[H_l, \quad L_A], E_\alpha] &= \sum_\beta a(\alpha A\beta) r_l(\alpha) \delta\{r_1(\alpha) - r_1(\beta)\} E_\beta \\ &+ \sum_m a(\alpha A m) r_l(\alpha) \delta\{r_1(\alpha)\} H_m \\ &- \sum_\beta b(lA_\beta) \delta\{r_1(\beta)\} [E_\beta, \quad E_\alpha] \\ &- \sum_m b(lAm) r_m(\alpha) E_\alpha .\end{aligned}$$

Comparing the terms in E_α in these two expressions, we get

$$\sum_m b(lAm) r_m(\alpha) = 0. \tag{6}$$

Since (6) is true for all α, and since the $r_m(\alpha)$ span an m-dimensional vector space as α varies over the allowed range, it follows that

$$b(lAm) = 0. \tag{7}$$

Two possible cases arise now.

(a) If H_1 is such that $r_1(\beta) \neq 0$ for any β it follows that[3]

$$[H_l, \quad L_A] = 0$$

and hence every element of S commutes with every element of L. Hence the statement of the theorem is trivially satisfied with

$$L_A^0 = L_A \cdot L_A^1 = 0.$$

(b) If there are some E_β for which $r_1(\beta) = 0$ we have only proved that

$$[H_l, \quad L_A] = \sum_\beta b(lA\beta) \delta\{r_1(\beta)\} E_\beta. \tag{8}$$

But making use of (2b) we can then show

$$\sum_\beta \{b(lA\beta) r_m(\beta) - b(mA\beta) r_l(\beta)\} \delta\{r_1(\beta)\} E_\beta = 0.$$

This relation entails the existence of numbers $q(A\beta)$, not necessarily nonzero, such that

$$b(lA\beta) = q(A\beta) r_l(\beta). \tag{9}$$

From (4), we can deduce

$$\begin{aligned}[][[E_\alpha, \quad E_\beta], L_A] &= \sum_\gamma a(\beta A\gamma) \delta\{r_1(\beta) - r_1(\gamma)\} [E_\alpha, E_\gamma] \\ &- \sum_\gamma a(\alpha A\gamma) \delta\{r_1(\alpha) - r_1(\gamma)\} [E_\beta, \quad E_\gamma]\end{aligned}$$

which implies

$$a(\alpha+\beta \ A \ \alpha+\beta) = a(\alpha A\alpha) + a(\beta A\beta).$$

This requires the existence of numbers $p(Al)$, not necessarily nonzero, such that

$$a(\alpha A\alpha) = \sum_l p(Al) r_l(\alpha). \tag{10}$$

Now consider the quantity

$$L_A^0 = L_A + \sum_l p(Al) H_l + \sum_\beta q(A\beta) E_\beta. \tag{11}$$

It is then immediately verified that

$$[L_A^0, \quad H_l] = 0. \tag{12}$$

Consequently, we make use of (4) and (11) to write

$$[L_A^0, \quad E_\alpha] = \sum_\beta e(\alpha A_\beta) E_\beta + \sum_m e(\alpha Am) H_m;$$

we can deduce

$$\sum_\beta e(\alpha A_\beta)\{r_m(\alpha) - r_m(\beta)\} E_\beta + \sum_l r_m(\alpha) e(\alpha Al) H_l = 0,$$

so that

$$e(\alpha A\beta) = 0; e(\alpha Al) = 0.$$

Hence

$$[E_\alpha, \quad L_A^0] = 0. \tag{13}$$

If we now write the commutation relations of the Lorentz algebra in the form

$$[L_A, \quad L_B] = \sum_C \gamma_{AB}^C L_C, \tag{14}$$

we can rewrite it in the form

$$\begin{aligned}[][L_A^0, \quad L_B^0] - \sum_C \gamma_{AB}^C L_C^0 &= [L_A - L_A^0, L_B - L_B^0] - \sum_C \gamma_{AB}^C (L_C - L_C^0).\end{aligned}$$

Since the expression on the left-hand side commutes with H_l, E_α while the right-hand side is linear in them, both sides must identically vanish; this gives the basic result

$$[L_A^0, \quad L_B^0] = \sum_C \gamma_{AB}^C L_C^0, \tag{15}$$

$$[L_A^1, \quad L_B^1] = \sum_C \gamma_{AB}^C L_C^1, \tag{16}$$

with

$$L_A^1 = L_A - L_A^0. \tag{17}$$

This concludes the demonstrations of the theorem.[4]

[4] After this work was completed, the author had the opportunity to learn that essentially the same results have been deduced by V. Ottoson, A. Kihlberg, and J. Nilsson, "Internal and Space–Time Symmetries," Phys. Rev. **137**, B658 (1965). See also L. Michel, Phys. Rev. **137**, B405 (1965).

We can deduce an important corollary from this theorem:

Corollary. In every unitary representation of the full symmetry algebra, L_A^1 must identically vanish; and hence we get a unitary representation of a direct sum of L and S only.

This follows since every unitary representation of the algebra S is the direct sum of irreducible finite dimensional unitary representations. Consequently, the quantities L_A^1 satisfying (16) have a unitary representation which is the direct sum of finite dimensional unitary representations. But the only such representations are trivial.[5]

We now make several remarks:

(1) The restriction to a simple group S can be easily relaxed to any semisimple group, the only requirement being that the generator Q must have nonvanishing "parts" in each of the simple algebras which occur in the direct sum decomposition of the semisimple algebra.

(2) The theorem is equally applicable if the Lorentz algebra L is replaced by any other Lie algebra, say the algebra of the Galilei group. In this case the corollary is no longer applicable since the Galilei group has nontrivial finite-dimensional unitary (nonfaithful) representations.

(3) We could interchange the roles of the internal symmetry algebra and the Lorentz algebra: if we require that any one element of the homogeneous Lorentz algebra, say M_{12}, commute with all elements of the symmetry algebra, then every element of the internal symmetry algebra could be expressed in the form

$$\begin{aligned} H_l &= H_l^0 + H_l^1, \\ E_\alpha &= E_\alpha^0 + E_\alpha^1, \end{aligned} \tag{18}$$

with H_l^1, E_α^1 being linear combinations of the elements of the Lorentz algebra, such that H_l^0, E_α^0 commute with every element of the Lorentz algebra.[6] We can then show these quantities satisfy the relations

$$\begin{aligned} [H_l^i,\ E_\alpha^k] &= \delta^{ik} r_l(\alpha) E_\alpha^i, \\ [H_l^i,\ H_m^k] &= 0, \\ [E_\alpha^i,\ E_{-\alpha}^k] &= \delta^{ik} \sum_l r_l(\alpha) H_l^i, \\ [E_\alpha^i,\ E_\beta^k] &= \delta^{ik} N_{\alpha\beta} E_{\alpha+\beta}^i, \end{aligned}$$

which generalize the commutation relations (2).

(4) In the demonstration above, H_l^1 and E_α^1 are linear combinations of those elements of L which commute with all elements of S. Hence if we require that the rotation subalgebra generated by M_{23}, M_{31}, M_{12} all commute with S, H_l^1 and E_α^1 must be linear combinations of the elements of L which are invariant under rotations. To see this we note that the decomposition (18) is unique since if

$$\begin{aligned} H_l &= H_l^0 + H_l^1 = H_l'^0 + H_l'^1, \\ E_\alpha &= E_\alpha^0 + E_\alpha^1 = E_\alpha'^0 + E_\alpha'^1, \end{aligned}$$

then

$$H_l^0 - H_l'^0 = H_l'^1 - H_l^1$$

and

$$E_\alpha^0 - E_\alpha'^0 = E_\alpha'^1 - E_\alpha^1$$

must commute with every element of L but they are at the same time elements of L. Hence they belong to the center of L, which is trivial. Hence H_l^1 and E_α^1 are unique and hence must be invariant under rotations. But the only element of L invariant under rotations is the Hamiltonian (time translation generator); consequently the commutator of any two elements of H_l^1, E_α^1 vanish which implies, by virtue of (19), that they themselves vanish. Hence if the elements of S commute with space rotations, we get only a trivial direct sum structure.

The present work in conjunction with that of other authors imply the extreme difficulty of constructing a purely Lie algebra model of an exact symmetry involving mass splittings. Any such scheme would require for its success a Lie algebra whose elements cannot be expressed as linear sums of elements of the internal symmetry algebra and the Lorentz algebra only.

Note added in proof: A definitive theorem in this connection has been proved in L. S. O'Raifeartaigh, Phys. Rev. Letters **14**, 575 (1965).

ACKNOWLEDGMENTS

The essential part of this work was done during a visit to the Argonne National Laboratory. It is a pleasure to thank Professor M. Peshkin for his hospitality at the Argonne National Laboratory and Professor N. Svartholm for the opportunity to visit the Chalmers University of Technology. Very useful discussions with Professor M. Hamermesh, Dr. W. D. McGlinn, Dr. Jan Nilsson, Dr. M. Y. Han, and Professor M. E. Mayer are gratefully acknowledged.

[5] It is interesting to point out that Ottoson, Kihlberg, and Nilsson (Ref. 4) have considered nonunitary representations of the Lorentz group, relating the nonunitary nature to the instability of several members of each multiplet.

[6] This result has been deduced by Y. Tomozawa, "Internal Symmetry and the Poincaré Group," J. Math. Phys. **6**, 656 (1965).