THE HARMONIES IN THE MOTION OF CELESTIAL BODIES UNDER NEWTONIAN ATTRACTION†

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I. Introduction

Invariance principles have often been known to be directly connected with fundamental and general principles underlying the dynamical framework, going beyond any particular dynamical model under consideration. Thus we are familiar with the rotation invariance and translation invariance of particle dynamics as the manifestation of the isotropy and the homogeneity of space respectively. We know that these invariance principles entail the conservation laws of angular momentum and of linear momentum for an isolated system. For isolated systems, the Hamiltonian must be invariant under rotations and translations. However, not all relevant invariance principles are invariances of the Hamiltonian: the principle of relativity is a fundamental invariance principle for the dynamics of an isolated system, but it does not leave the Hamiltonian of the system unaltered. We should thus be interested not only in invariance principles which lead to invariance groups of the Hamiltonian but more general invariances of the dynamical system.

These two classes of invariance principles have their applications in particle physics. In the first instance we are interested in the groups which leave the Hamiltonian invariant. These lead to degeneracies of the (energy) eigenstates and may be associated with the "observed" multiplet structure of the particles. To the extent that the multiplets have approximately equal masses, we have to consider the invariance group only to be an "approximate invariance." We may also consider groups which do not leave the masses equal, but account for a mass spectrum.

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In these discussions it is to be noted that if we discuss the structure of a system of two particles in interaction in its center of mass frame, it can be reduced to an equivalent one-body problem with an "external" potential. The energy levels of the equivalent one-body problem correspond to "masses" for the two-body problems; similarly, the angular momentum in the one-body problem corresponds to the "spin" of the composite system. Energy level degeneracy in the one-body problem is the analogue of the mass degeneracy for the composite system. Thus, in the discussion of the hydrogen atom, for example, we are really talking about the mass-spin spectrum of a family of particles, but it is more convenient to talk of the energy and angular momentum spectrum of a particle under Newtonian attraction.

Returning now to the invariance group and degeneracy of energy levels of a particle in a potential well, we often encounter a degeneracy going beyond the familiar magnetic quantum number degeneracy associated with rotational invariance. This is often referred to as "accidental degeneracy," though more detailed study shows that there is little accident involved in the occurrence of this additional degeneracy. Among the elementary solvable systems of quantum mechanics, the rigid rotator exhibits no accidental degeneracy, but the isotropic oscillator and the hydrogen atom exhibit accidental degeneracies. We will see the dynamical reason behind this occurrence for the hydrogen atom. While some work has been initiated on the problem of degeneracies in general quantum mechanical systems, the topic deserves further study.[1)]

II. The Stationary States of the Hydrogen Atom

Let us now consider a two-particle system consisting of two particles with masses m_1 and m_2 with the Hamiltonian

$$\mathcal{H} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{G}{|\underline{q}_1 - \underline{q}_2|}$$

which could be rewritten in terms of center-of-mass and relative variables according to

$$\mathcal{H} = \frac{\underline{p}^2}{2M} + \frac{\underline{p}^2}{2\mu} - \frac{G}{q} .$$

Since the Hamiltonian is invariant under rotations (in the relative motion variables), the angular momentum is a conserved quantity in the equivalent one-body problem with the Hamiltonian

$$\mathcal{H} = \frac{\underline{p}^2}{2\mu} - \frac{G}{q} .$$

The stationary states of the system are labelled by three quantum numbers n_r, ℓ, m. The stationary states are given by the respective wavefunctions

$$\langle r\theta\phi | n_r, \ell, m\rangle = Y_\ell^m(\theta,\phi) R_{n_r,\ell}(r)$$

$$R_{n_r,\ell}(r) = \sqrt{\frac{(n_r-1)!G}{(n_r+\ell)^2[(n_r+2\ell)!]^3}} \left(\frac{2Gr}{n_r+\ell}\right)^{\ell+1} e^{-Gr/(n_r+\ell)} L_{n_r-1}^{\ell+1}\left(\frac{2Gr}{n_r+\ell}\right).$$

where

$$L_N^{\nu}(x) = \frac{d^\nu}{dx^\nu}\left\{e^x \frac{d^N}{dx^N}(x^N e^{-x})\right\} .$$

The units are so chosen that $\mu = 1$, $\hbar = 1$. The eigenvalues of the Hamiltonian are given by

$$E_n = -\frac{G^2}{2(n_r+\ell)^2} = -\frac{G^2}{2n^2}; \qquad n = n_r + \ell$$

$$E_1 = -\frac{G^2}{2} .$$

In this form we see the "accidental" degeneracy: the energy does n depend upon both n_r and ℓ separately but only through the principa quantum number $n = n_r + \ell$. In the conventional spectroscopic notati we have the 1S state nondegenerate; 2S and 2P levels are degener ate with each other; 3S, 3P, and 3D levels are degenerate; and so c

This accidental degeneracy can be ascribed to the existence of a suitable group of invariance of the Hamiltonian.[2)] This group must have the rotation group O(3) as a subgroup. Such a group is given by O(4). An irreducible representation of O(4), when restricte to its O(3) subgroup, becomes reducible. A special class of the (unitary irreducible) representations of O(4) (the "symmetric tensor representations") has the property that every (one-valued) represen tation of O(3) up to a maximum angular momentum ℓ and occurs onc (and only once).

III. The Invariance Group

The degeneracy and the corresponding invariance group mus be associated with an additional set of conservation laws. From ro tational invariance itself we have

$$[L_{ab}, H] = 0 \qquad L_{ab} = q_a p_b - q_b p_a .$$

We can show further that there exists a constant of motion, called the Lenz vector,[3] which satisfies the commutation relations

$$\left[(q\cdot p)p_a + ((G/q) - p^2)q_a\, ,\ H\right] = 0$$

by direct computation. This quantity is not Hermitian; it could be made Hermitian by a straightforward rearrangement of factors. We define[4]

$$A_a = \frac{1}{\sqrt{-2H}}\left\{ \tfrac{1}{2}p_a\,(q\cdot p) + \tfrac{1}{2}(q\cdot p)p_a + \frac{G}{q}\,q_a - \tfrac{1}{2}p^2 q_a - \tfrac{1}{2}\,q_a p^2 \right\}.$$

Then

$$[A_a, H] = 0$$

so that, like L_{ab}, A_a is also conserved. On the other hand we may view L_{ab} and A_a as generators of unitary transformations that leave H invariant. In their capacity as generators the commutation relations satisfied by them are of interest. We already know that L_{ab} satisfy the commutation relations

$$[L_{ab}, L_{cd}] = i\left(\delta_{bc}L_{da} + \delta_{ad}L_{cb} - \delta_{ac}L_{db} - \delta_{bd}L_{ca}\right)$$

appropriate to the Lie algebra of O(3). The additional relations[4]

$$[L_{ab}, A_c] = i\left(\delta_{ac}A_b - \delta_{bc}A_a\right);\ [A_a, A_b] = iL_{ab}$$

can be easily verified. These are easily identified to be the commutation relations of the group O(4), especially if we define:

$$L_{a4} \equiv A_a$$

and consider

$$L_{ab}, \qquad a, b = 1, 2, 3, 4$$

as the generators of O(4). With this particular construction of O(4) it is easy to see that

$$\epsilon_{abc}L_{ab}A_c = \epsilon_{abcd}L_{ab}L_{cd} = 0.$$

The O(4) has two Casimir invariants, $L_{ab}L_{ab}$ and $\epsilon_{abcd}L_{ab}L_{cd}$. Whenever the second Casimir invariant vanishes we have a symmetric tensor representation. The Bargmann construction used above thus generates only symmetric tensor representations. A symmetric tensor representation of O(4) of rank ν splits up into the sum of $\nu+1$ representations of O(3) of ranks ℓ satisfying $\nu \geqq \ell \geqq 0$ and thus has

$$\sum_{\ell=0}^{\nu} (2\ell+1) = (\nu+1)^2.$$

Not only does the Fock-Bargmann O(4) symmetry group account for the accidental degeneracy (and entail the existence of the Lenz vector which is a constant of motion), but they account for the complete spectrum and lift the entire degeneracy. That is, the O(4) labelling uniquely specifies the state. In particular, the Hamiltonian itself is an invariant of the O(4) group given by

$$H = -2(L_{ab}L_{ab})^{-1}.$$

We also observe that the generalized enveloping algebra of O(4) (that is, the associative algebra generated by the six generators and their analytic functions) is the maximal subalgebra which commutes with the Hamiltonian. In other words, the (generalized) enveloping algebra of O(4) is the "commutant" of the Hamiltonian.

The Fock-Bargmann choice is the simplest choice for the degeneracy group of the stationary states of the hydrogen atom. We note, however, that the O(4) group is a "parity-mixing" group, since it lumps together states with even and odd orbital angular momenta. We could, if we so chose, construct a symmetry group which organizes all the positive parity states of the same energy into a single irreducible multiplet and all the negative parity states into another irreducible multiplet. The relevant group is SU(3) and the relevant representations are the symmetric tensor representations. The SU(3) group has 3^2-1 generators of which $3\cdot 2/1\cdot 2$ are given by the generators of O(3). The other $(3\cdot 4/1\cdot 2)-1$ generators can be constructed in terms of analytic functions of the angular momentum and bilinear quantities in the Lenz vector. (The construction is given elsewhere.)[5] No <u>new</u> conservation law is implied; the conservation of the generators of SU(3) implies the conservation of certain bilinear quantities in the Lenz vector, and is automatically implied by the conservation of the Lenz vector and of the angular momentum. The SU(3) group, however, does not account completely for the degeneracy, since (except for the ground state) <u>two</u> (irreducible, unitary) SU(3) representations are degenerate in energy and are needed to fully characterize the spectrum. The (generalized) enveloping algebra is a proper subalgebra of the commutant of the Hamiltonian. The commutant can be generated by adjoining to the enveloping algebra of SU(3) the Lenz vector itself; this has the requisite odd parity to intertwine two SU(3) multiplets with opposite parity.

Even within the choice of the O(4) group, there exists other constructions which, while not as straightforward as the Bargmann construction, also account for the ℓ degeneracy of the states. The general O(4) group is specified by two parameters which specify the highest and lowest ℓ values allowed within the representation. For the special choice of these limits to be $n-1$ and 0 respectively it coincides with the Bargmann construction. For these more general objects the second Casimir operator $\epsilon_{abcd}L'_{ab}L'_{cd}$ does not vanish; consequently not all states are included. Again, no new conservation law is implied apart from the conservation of the Lenz vector. We will return to the choice of this O(4) group later.

The discussion so far dealt with the negative energy states. Strictly speaking, there are no positive energy stationary states since the spectrum is continuous. One may nevertheless consider the group generated by the Lenz vector and the angular momentum; the generalized enveloping algebra is still the commutant of the Hamiltonian. The generators continue to satisfy the commutation relations of O(4). Considered as generating canonical transformations we see, however, that the generators A_a are pure imaginary. We could thus define $\mathcal{A}_a$ by

$$\mathcal{L}_{a4} = \mathcal{A}_a = -\mathcal{L}^{a4} = \frac{1}{\sqrt{2H}}\left\{\tfrac{1}{2}p_a(q\cdot p) + \tfrac{1}{2}(q\cdot p)p_a + \frac{G}{q}q_a - \tfrac{1}{2}p^2 q_a - \tfrac{1}{2}q_a p^2\right\}$$

which together with $\mathcal{L}^{ab} = \mathcal{L}_{ab}$ where, $1 \leqq a, b \leqq 3$, constitute an O(3,1) group[4] (the 3+1 Lorentz group). All its unitary representations are infinite dimensional (except the trivial one-dimensional representation), which corresponds to the fact that for <u>any</u> positive energy (ideal) eigenstate, <u>all</u> angular momenta are allowed. The Hamiltonian is still given in terms of the Casimir operator according to

$$H = -2(\mathcal{L}^{ab}\mathcal{L}_{ab})^{-1}.$$

The second Casimir operator vanishes for the Bargmann choice of O(3,1), but is in general nonzero. The analogue of the SU(3) generators could also be constructed. We note, in passing, that the discrete and continuous nature of the energy spectrum corresponds to the discrete and continuous nature of the eigenvalues of the Casimir operators of O(4) and of O(3), respectively.

The theory for both positive and negative energy states could be generalized to the case of arbitrary number N of space dimensions.[5] In general there is an O(N) degeneracy from rotational invariance, and an accidental degeneracy which escalates the symmetry to be $O(N+1)$. One could also split it up into even and odd rank symmetric tensor representations of $O(N)$, each set constituting an irreducible representation of $SU(N)$. For the positive energy states we could associate the degeneracy with $O(N,1)$.

In the above expressions G serves the purpose of a coupling constant. For $G \to \infty$ we have the strong coupling limit; in the case $|\underline{A}| \to \infty$ also, so we should define the renormalized quantities $G^{-\frac{1}{2}} \underline{A}$ for the generators. It is then seen that in the strong coupling limit these generators commute and we have as the symmetry group the Euclidean group $E(3)$ in three dimensions. Each level is now infinitely degenerate (since the principal quantum number increases without limit for any finite negative energy as $G \to \infty$). The other extreme $G \to 0$ gives the case of the free particle; in this case the generators are finite as they stand. But there is no negative energy spectrum. For the positive energy spectrum the $O(3,1)$ group continues to exist as before. Incidentally, both in the $G=0$ and $G \neq 0$ cases the group $O(3,1)$ can be "contracted" to yield $E(3)$ if we so choose.

IV. The Non-Invariance Group

The invariance groups of the Hamiltonian are analogous to the invariance groups of particle physics. But in particle physics another fundamental item of information concerns the occurrence and multiplicity of the various multiplets. Thus, for example, the occurrence of two isospin doublets with hypercharges ± 1 and of an isospin triplet and an isospin singlet with hypercharge zero amongst the baryons of the same spin and parity was crucial in the synthesis of these particles into an octet under $SU(3)$. Similarly, the existence of various $O(4)$ representations amongst the bound states of the hydrogen atom contains additional dynamical information. Could we give a group-theoretic characterization of this information? We know that the bound state spectrum contains each symmetric tensor representation of $O(4)$ once and only once. Could we organize these multiplets into a single irreducible multiplet of a larger group? This is in complete parallel to the organization of $O(3)$ multiplets to construct an $O(4)$ symmetric tensor representation. We can thus verify[5)] that the set of all symmetric tensor representations of $O(4)$ up to the ν^{th} rank symmetric tensor constitutes the irreducible symmetric tensor representation of rank ν of the group $O(5)$. In contrast to the $O(4)$ group which was an invariance group of the Hamiltonian, the $O(5)$ group contains transformations which take states with one energy into states with another energy. However, the states with energies $-(R/n^2)$, $1 \leqq n \leqq \nu+1$ form a closed system.

The $O(5)$ group has $5 \cdot 4/1 \cdot 2$ generators of which $4 \cdot 3/1 \cdot 2$ generators are given by the $O(4)$ generators. The additional four generators transform as a four-vector with regard to $O(4)$. The commutation relations are given by the standard rotation group commutation relations:

$$\frac{1}{i}[J_{ab}, J_{cd}] = \delta_{bc}J_{da} - \delta_{ac}J_{db} + \delta_{ad}J_{cb} - \delta_{bd}J_{ca}; \quad 1 \leq a,b,c,d \leq 5.$$

There are two Casimir invariants—the first one is $J_{ab}J_{ab}$. To define the second one we introduce the five-vector

$$W_a = \frac{1}{8}\epsilon_{abcde}J_{bc}J_{de}$$

and construct the invariant W_aW_a. We note that W_5 is the second Casimir invariant of the O(4) subgroup; for the Bargmann O(4) it vanishes and hence any O(5) generators which include the Bargmann O(4) generators would lead to vanishing second Casimir invariant. If we denote the generators of O(3) by $\underline{L}$, the additional O(4) generator by $\underline{A}$ and the additional O(5) generators by S, $\underline{B}$ we have the restrictions

$$\underline{A}\cdot\underline{L} = 0; \quad \underline{B}\cdot\underline{L} = 0; \quad A_jB_k - A_kB_j = \epsilon_{jk\ell}L_\ell S.$$

$$Q = \underline{L}^2 + \underline{A}^2 + \underline{B}^2 + S^2 = \nu(\nu+3).$$

Instead of considering a finite set of O(4) representations, it is also possible to organize all the O(4) representations (or all except the first n representations) into a single irreducible multiplet of the noncompact group O(4,1) (the 4+1 de Sitter group). The generators are given by $\underline{B} \to \underline{B}' = i\underline{B}$; $S \to S' = iS$. There are two classes of O(4,1) representations with the second Casimir invariant zero.[6] The first corresponds to

$$-Q = \{\underline{L}^2 + \underline{A}^2 - \underline{B}'^2 - S'^2\} = \mu^2 + \frac{9}{4} \qquad (\mu \text{ integral})$$

which contain all the symmetric tensor representations of O(4). The second class corresponds to

$$-Q = (\lambda - 1)(\lambda + 2) = 9/4 - (\lambda + 3/2)^2, \quad \lambda = 1,2,3,\ldots$$

which includes only those states with $\nu = n-1 \geq \lambda$. The matrix representation of the O(4,1) generators are known and are the analytic continuation of the O(5) representations. The choice of the parameter λ is at out disposal: we thus have a "sliding" group,[7] the sliding index given by λ.

If instead of choosing the Bargmann O(4) we chose a more general O(4), we would have to consider other kinds of representations of O(4,1). For not every possible choice of O(4) could we organize the corresponding O(4) representations into an O(4,1) representation. If we choose the O(4) representations to contain only those states with $n-1 \geqq \ell \geqq s$, those O(4) representations with s fixed and $\infty > n > s$ can be fitted into a representation of O(4, 1) with

$$Q = -2(s^2 - 1);$$

$$+W_1^2 + W_2^2 + W_3^2 + W_4^2 - W_5^2 = -s^2(s^2-1).$$

Related considerations apply to the case of the hydrogen atom in an arbitrary number of dimensions. The construction in terms of the symmetric tensors generalize immediately. The singular case of the atom in one dimension is particularly amusing. In this case the negative energy levels are nondegenerate and the noninvariance group is O(2, 1) (the 2 + 1 Lorentz group). The reciprocal of the Hamiltonian is the 1 - 2 generator in the case of negative energy states and is thus discrete. But for the positive energy states the character of the group changes to O(1, 2), which is isomorphic to O(2, 1) but with the generators identified differently. The 1 - 2 generator now has a continuous spectrum corresponding to the continuous positive energy spectrum.

V. Group Theory in Celestial Dynamics

The group-theoretic structure could equally well be discussed within the framework of the classical theory[8] of the Newtonian potential. The energy "spectrum" is now continuous for the negative energy states also, in the sense that there are bound states of any (negative or positive) energy. The O(3) degeneracy now corresponds to elliptic orbits of the same size and shape, but oriented differently. The O(3) group is thus kinematical. To discuss the group structure we replace the commutator by i times the classical Poisson brackets. The O(4) degeneracy is now associated with the existence of additional canonical transformations which change the shape (the eccentricity) of the elliptic orbit while preserving its energy. For any fixed energy $E < 0$, the square of the angular momentum ℓ^2 ranges from 0 to $-1/2E$, i.e., from the straightline motion to circular motion. For a given ℓ^2 and E, the eccentricity is given by

$$e = \sqrt{1 + 2E\ell^2}.$$

For given e and ℓ^2, the point P furthest from the focus S (the aphelion) is at a distance

$$q_o = \frac{\ell^2}{1 - e}.$$

The Lenz vector $\underline{\Lambda} = (q \cdot p)\, \underline{p} + ((1/q) - p^2)\underline{q}$ has the magnitude e and points always from S to P. Under an infinitesimal canonical transformation generated by $\underline{\epsilon} \cdot \underline{\Lambda}$ the parameters of the orbit change according to:

$$\begin{aligned} \delta E &= 0 \\ \delta \underline{L} &= -\underline{\epsilon} \times \underline{\Lambda} \\ \delta \underline{\Lambda} &= 2E\ \underline{\epsilon} \times \underline{L} \\ \delta e &= -(1-e^2)\, \underline{\epsilon} \cdot \hat{d} \end{aligned}$$

where $\hat{\underline{d}}$ is the unit vector in the direction $\underline{L} \times \underline{\Lambda}$. Those formulae continue to apply as the energy rises (and the eccentricity increases) to reach parabolic $(e = 1)$ and hyperbolic $(e > 1)$ orbits.

The noninvariance groups can also be studied within the classical framework[8] where Poisson brackets again replace (-i times) the commutator bracket. The O(5) and O(4,1) groups now change an orbit into another orbit which, in general, has a different energy. The explicit form of the generators can be obtained either by considering the limiting form of the quantum mechanical generators or, more directly, by the solution of the differential equations entailed by the Poisson bracket relations realizing the O(4,1) group structure. Using the latter method Bacry[9] has worked out the general solution, he obtains

$$S = -\sqrt{a - 1/2H}\left\{ (q \cdot p\sqrt{-2H}\ \sin((q \cdot p\sqrt{-2H} + \Theta) + (1 + 2Hq)\cos((q \cdot p)\sqrt{-2H} + \Theta) \right\}$$

$$\underline{B} = \sqrt{a - 1/2H}\left\{ q\sqrt{-2H}\ \cos((q \cdot p)\sqrt{-2H} + \Theta)\,\underline{p} - (q \cdot p)\sin((q \cdot p)\sqrt{-2H} + \Theta)\,\underline{p} + 1/q\ \sin((q \cdot p)\sqrt{-2H} + \Theta)\,\underline{q} \right\}$$

where a is a (real) parameter and Θ is a real arbitrary function of H. It is straightforward to show that Θ simply reflects the freedom to make energy-dependent canonical transformations: these would leave $\underline{L}$ and $\underline{A}$ unaltered but would change S and $\underline{B}$ in form. By direct calculation we can show that

$$[\phi(H), S] = (-2H)^{3/2} \frac{\partial \phi}{\partial H} \frac{\partial S}{\partial \Theta}$$

Thus by a proper canonical transformation we can eliminate Θ altogether. The $1/a$ cannot be so eliminated. We see that for all quite different in a theory of elementary particles; here the "observed" properties could find their simplest expression in terms of groups. The (canonical?) dynamical variables could then be secondary constructs. The real question concerns the possibility of describing interaction between two systems, and whether the description of the interaction between two such systems finds its simplest expression in the canonical framework or in the non-invariance enveloping algebra framework.[10)] The systems which have a simple description of interaction in one form are bound to have complicated description in the other form (and vice versa) in view of the complicated relation between the dynamical variables and the generators. Already some examples of nontrivial dynamical schemes in terms of a non-invariance group are known. These questions deserve further study.

In any case, the question of the degeneracies and spectra of quantum mechanical systems provide, in themselves, a fascinating topic of study.[1)]

Since these lectures were given we have heard of the tragic death of Professor H. J. Bhabha in an air crash near Geneva. The authors dedicate this paper as a token of esteem and admiration to the memory of Homi Jehangir Bhabha, physicist, teacher, educator, art critic, engineer, statesman and diplomat.

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