PHYSICAL REVIEW VOLUME 157, NUMBER 5 25 MAY 1967

Dynamics of Coherent States*

C. L. MEHTA
Department of Physics and Astronomy, University of Rochester, Rochester, New York

AND

P. CHAND,† E. C. G. SUDARSHAN, AND R. VEDAM‡
Physics Department, Syracuse University, Syracuse, New York

(Received 6 October 1966)

A system of harmonic oscillators in the presence of interaction, and with an arbitrary number of degrees of freedom, is considered. The most general form of the Hamiltonian is derived under the restriction that the states which are initially coherent remain coherent at all times. The equation of motion for the annihilation operator, obtained by using this Hamiltonian, is solved, and the frequency spectrum of the annihilation operator is discussed. By giving specific examples it is shown that, in general, the annihilation operators (or their eigenvalues) contain positive as well as negative frequency components and hence are not analytic signals. Some special cases are also considered where the annihilation operators are analytic signals.

I. INTRODUCTION

SEVERAL publications have appeared in recent years dealing with the properties and applications of coherent states.[1–7] These states are defined as the eigenstates of the annihilation operator and are analogous to the classical deterministic situation. Since the annihilation operator is not Hermitian, the eigenvalues are, in general, complex and the states belonging to different eigenvalues are not orthogonal. However, these states form a complete set and can be used as a basis for expanding arbitrary states and arbitrary operators. In this connection, they have been found to be very useful in the description of optical coherence for free-electromagnetic fields.[4,6,8,9] It is generally known that for free fields the time-dependent annihilation operators have only *positive-frequency* components so that the eigenvalue of any linear combination of such operators is an analytic signal.[10] It is of interest to study the time dependence of the eigenvalues of the annihilation operators in the presence of interaction. In a recent paper, Glauber[11] has shown that if the time derivative of the annihilation operator does not involve a functional dependence on the creation operator, i.e., if

$$d\hat{a}(t)/dt = f(\hat{a}(t),t)\,, \tag{1.1}$$

then the states which are initially coherent remain coherent at all times. In another paper[12] we have shown that the requirement (1.1) is both necessary and sufficient for the states to remain coherent at all times and that for a system described by a (Hermitian) Hamiltonian, the function f must be linear in the annihilation operators. We have also obtained the general form of the Hamiltonian consistent with this requirement, and showed that the eigenvalues of the annihilation operator are in general not analytic signals. In Ref. 12, however, only systems with one degree of freedom were considered. In the present paper some of these results are generalized for systems with an arbitrary number of degrees of freedom.

In Sec. II we consider a system of harmonic oscillators with arbitrary number of degrees of freedom and derive the general form of the Hamiltonian with the requirement that the states which are initially coherent remain coherent at all times. In Sec. III the equation of motion for the annihilation operator is solved and Sec. IV deals with the frequency spectrum of the annihilation operator. In the Appendix we derive the general form of a unitary matrix $U(t)$ which is an analytic signal and is such that $U(t)$ and $U(t')$ commute for all t and t'.

II. TIME EVOLUTION OF COHERENT STATES: CONDITIONS ON THE HAMILTONIAN

Let us consider a system of harmonic oscillators which is described by the canonical operators[13] $\hat{q}_\lambda$ and $\hat{p}_\lambda$ satisfy the commutation relations[14]

$$[\hat{q}_\lambda,\hat{p}_{\lambda'}]=i\delta_{\lambda\lambda'}\,;\quad [\hat{q}_\lambda,\hat{q}_{\lambda'}]=[\hat{p}_\lambda,\hat{p}_{\lambda'}]=0\,. \tag{2.1}$$

The system can equally be described in terms of the canonical annihilation and creation operators $\hat{a}_\lambda$ and

* This research was supported in part by the U. S. Army Research Office (Durham) and by the U. S. Atomic Energy Commission.

† Present address: A & M University, College Station, Texas.
‡ Permanent address: Physics Department, Harvard University, Cambridge, Massachusetts.

1 J. Schwinger, Phys. Rev. **91**, 728 (1953).
2 J. R. Klauder, Ann. Phys. (N. Y.) **11**, 123 (1960).
3 R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963).
4 E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
5 R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
6 C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).
7 J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. **6**, 733 (1965).
8 R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
9 L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).
10 The concept of analytic signal was introduced by D. Gabor, J. Inst. Elect. Engrs. (London) **93**, 429 (1946); see also Ref. 9, p. 236.
11 R. J. Glauber, Phys. Letters **21**, 650 (1966).
12 C. L. Mehta and E. C. G. Sudarshan, Phys. Letters **22**, 574 (1966).
13 In this paper we denote all operators by a circumflex, e.g., $\hat{q}$, $\hat{a}$, $\hat{a}^\dagger$, etc.
14 We have chosen units such that $\hbar=1$.

$\hat{a}_\lambda{}^\dagger$ defined by

$$\hat{a}_\lambda=\frac{1}{(2\omega_\lambda)^{1/2}}(\omega_\lambda\hat{q}_\lambda+i\hat{p}_\lambda)\,, \tag{2.2}$$

$$\hat{a}_\lambda{}^\dagger=\frac{1}{(2\omega_\lambda)^{1/2}}(\omega_\lambda\hat{q}_\lambda-i\hat{p}_\lambda)\,, \tag{2.3}$$

ω_λ being the frequency of the oscillator λ. The operators $\hat{a}_\lambda$ and $\hat{a}_\lambda{}^\dagger$ satisfy the commutation relations

$$[\hat{a}_\lambda,\hat{a}_{\lambda'}{}^\dagger]=\delta_{\lambda\lambda'}\,;\quad [\hat{a}_\lambda,\hat{a}_{\lambda'}]=[\hat{a}_\lambda{}^\dagger,\hat{a}_{\lambda'}{}^\dagger]=0\,. \tag{2.4}$$

The normalized coherent states which are the right eigenstates of the annihilation operator $\hat{a}_\lambda$ are then given by[2,5]

$$|\{v\}\rangle\equiv\prod_\lambda|v_\lambda\rangle=\prod_\lambda \exp\{v_\lambda\hat{a}_\lambda{}^\dagger-v_\lambda{}^*\hat{a}_\lambda\}|0\rangle\,, \tag{2.5}$$

and satisfy the relations

$$\hat{a}_\lambda|\{v\}\rangle=v_\lambda|\{v\}\rangle\,, \tag{2.6}$$

$$\hat{a}_\lambda{}^\dagger|\{v\}\rangle=\left\{e^{-\frac{1}{2}|v_\lambda|^2}\frac{\partial}{\partial v_\lambda}e^{\frac{1}{2}|v_\lambda|^2}\right\}|\{v\}\rangle\,. \tag{2.7}$$

Here the symbol $\{v\}$ is used to denote the sequence of complex numbers $v_1, v_2,\cdots v_\lambda,\cdots$ (v_λ being the eigenvalue of $\hat{a}_\lambda$); $|0\rangle$ denotes the vacuum (lowest energy) state and $\partial/\partial v_\lambda$ denotes formal partial differentiation with respect to v_λ keeping $v_\lambda{}^*$ and all other variables fixed.

To study the time evolution of the system, one can work either in the Heisenberg or in the Schrödinger picture. In the Heisenberg picture, the state is fixed and we consider the time evolution of the system in terms of the time evolution of the operators. The operator $\hat{a}_\lambda$ satisfies the Heisenberg equation of motion

$$i\frac{d\hat{a}_\lambda(t)}{dt}=[a_\lambda(t),\hat{H}(t)]\,, \tag{2.8}$$

and the problem then is to find how the state $|\{v(0)\}\rangle$ behaves in relation to the time-dependent operator $\hat{a}_\lambda(t)$. In particular we are interested in finding the conditions under which the state $|\{v(0)\}\rangle$ is an eigenstate of $\hat{a}_\lambda(t)$ for every λ and for all times t with an eigenvalue $v_\lambda(t)$. For very small τ, we can write, using (2.8),

$$\hat{a}_\lambda(t+\tau)=\hat{a}_\lambda(t)-i\tau[\hat{a}_\lambda(t),\hat{H}(t)]+O(\tau^2)\,. \tag{2.9}$$

If the state $|\{v(0)\}\rangle$, which is an eigenstate of $\hat{a}_\lambda(t)$ with an eigenvalue $v_\lambda(t)$, be also an eigenstate of $\hat{a}_\lambda(t+\tau)$ with an eigenvalue $v_\lambda(t+\tau)$, we must have

$$\begin{aligned}\hat{a}_\lambda(t+\tau)|\{v(0)\}\rangle&\\ &=v_\lambda(t+\tau)|\{v(0)\}\rangle\\ &=\left\{v_\lambda(t)+\tau\frac{\partial v_\lambda(t)}{\partial t}+O(\tau^2)\right\}|\{v(0)\}\rangle\,. \end{aligned}\tag{2.10}$$

Using also (2.9) and retaining only the terms which are of the first order in τ, we obtain the following eigenvalue relation

$$[\hat{a}_\lambda(t),\hat{H}(t)]|\{v(0)\}\rangle=i\frac{\partial v_\lambda(t)}{\partial t}|\{v(0)\}\rangle\,. \tag{2.11}$$

Since the states $|\{v(0)\}\rangle$ (which form a complete set) are simultaneous eigenstates of the operator $\hat{a}_\mu$ and the commutator $[\hat{a}_\lambda,\hat{H}]$, we must have

$$[\hat{a}_\mu,[\hat{a}_\lambda,\hat{H}]]=\frac{\partial}{\partial\hat{a}_\mu{}^\dagger}[\hat{a}_\lambda,\hat{H}]=0,\ \text{for all }\mu. \tag{2.11$'$}$$

This implies that the commutator $[\hat{a}_\lambda(t),\hat{H}(t)]$ must depend on the annihilation operators $\{\hat{a}(t)\}$ alone and not on any of the creation operators $\hat{a}_\mu{}^\dagger$, i.e., that

$$[\hat{a}_\lambda(t),\hat{H}(t)]=f_\lambda(\{\hat{a}(t)\},t)\,, \tag{2.12}$$

where f_λ is some function of the set of annihilation operators $\{\hat{a}\}$ and may also depend explicitly on t. Since Eq. (2.12) is valid for every λ, we note that $\hat{H}$ can at most be linear in the creation operators, i.e., $\hat{H}$ is of the form

$$\hat{H}=\sum_\lambda \hat{a}_\lambda{}^\dagger f_\lambda(\{\hat{a}\},t)+g(\{\hat{a}\},t)\,, \tag{2.13}$$

where g is some other function. Further since $\hat{H}$ is Hermitian, we see on taking the Hermitian adjoint of (2.13) that $\hat{H}$ is also at most, linear in the annihilation operators $\{\hat{a}\}$. Hence we conclude that $\hat{H}$ is of the form

$$\begin{aligned}\hat{H}=&\sum_\lambda\sum_\mu \omega_{\lambda\mu}(t)\hat{a}_\lambda{}^\dagger(t)\hat{a}_\mu(t)\\ &+\sum_\lambda\{F_\lambda(t)\hat{a}_\lambda{}^\dagger(t)+F_\lambda{}^*(t)\hat{a}_\lambda(t)\}+\beta(t)\,,\end{aligned} \tag{2.14}$$

where F_λ is some arbitrary function, the matrix $\omega_{\lambda\mu}(t)$ is Hermitian and $\beta(t)$ is real, i.e., that

$$\omega_{\lambda\mu}(t)=\omega_{\mu\lambda}(t)^*;\quad \beta(t)=\beta(t)^*. \tag{2.15}$$

We have thus shown that the necessary condition on the form of the Hamiltonian consistent with the requirement that the states which are initially coherent remain coherent at all times is given by (2.14). Though the condition for sufficiency is also built in the proof, one can see directly that if the Hamiltonian is of the form given by (2.14) the states which are initially coherent remain coherent at all times.[15]

The same conclusion can be obtained by working in the Schrödinger picture. In this case the operators $\{\hat{a}\}$ and $\{\hat{a}^\dagger\}$ are fixed but the state changes. The time development of the state is governed by the Schrödinger equation

$$i\frac{d}{dt}|\{v(t)\}\rangle=\hat{H}|\{v(t)\}\rangle\,. \tag{2.16}$$

[15] See for example Ref. 11.

Let us consider the case when $|\{v(t)\}\rangle$ is an eigenstate of $\hat{a}_\lambda$ with an eigenvalue $v_\lambda(t)$ for every λ and for all t. Again for very small τ, we can write, using (2.16),

$$|\{v(t+\tau)\}\rangle=|\{v(t)\}\rangle-i\tau\hat{H}|\{vt)\}\rangle+O(\tau^2). \quad (2.17)$$

Since we require that $|\{v(t+\tau)\}\rangle$ is an eigenstate of $\hat{a}_\lambda$ with eigenvalue $v_\lambda(t+\tau)$, we also have

$$\hat{a}_\lambda|\{v(t+\tau)\}\rangle = \left\{v_\lambda(t)+\tau\frac{\partial v_\lambda(t)}{\partial t}+O(\tau^2)\right\}|\{v(t+\tau)\}\rangle. \quad (2.18)$$

If we now use (2.17) and retain only the terms which are of first order in τ, we obtain the eigenvalue relation

$$[\hat{a}_\lambda,\hat{H}]|\{v(t)\}\rangle=i\frac{\partial v_\lambda(t)}{\partial t}|\{v(t)\}\rangle. \quad (2.19)$$

Hence, following an argument similar to that given in connection with the Heisenberg picture, it is readily seen that the general form of the Hamiltonian consistent with the requirement that the states which are initially coherent remain coherent at all times is given by[16]

$$\hat{H}=\sum_\lambda\sum_\mu \omega_{\lambda\mu}(t)\hat{a}_\lambda^\dagger\hat{a}_\mu +\sum_\lambda\{F_\lambda(t)\hat{a}_\lambda^\dagger+F_\lambda^*(t)\hat{a}_\lambda\}+\beta(t), \quad (2.20)$$

where F_λ is some arbitrary function, the matrix $\omega_{\lambda\mu}(t)$ is Hermitian and $\beta(t)$ is real, i.e., they satisfy Eq. (2.15).

It is interesting to note that if one expresses the Hamiltonian in terms of the variables $(\hat{q}_\lambda,\hat{p}_\lambda)$ then the form (2.14) or (2.20) is at most quadratic in $\hat{q}_\lambda$ and $\hat{p}_\lambda$. For such systems the dynamical brackets of the Wigner-Moyal[17] phase-space formulation of quantum mechanics (in which the Weyl's rule of association between operators and functions is used) reduce to Poisson brackets, so that quantum and classical equations of motion are identical.[18,19] If the form of the Hamiltonian is given by (2.14) or (2.20), a similar result holds even if one is using the phase-space formulation when the rule of association between operators and functions is that of normal ordering.[19]

[16] Apart from the c-number term $\beta(t)$, this form of the Hamiltonian has been noted as an example by Glauber (Ref. 11). However, as is shown here, this is the most general form of the Hamiltonian, consistent with the requirement that the states which are initially coherent, remain coherent at all times.

[17] (a) E. P. Wigner, Phys. Rev. **40**, 749 (1932); (b) J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949).

[18] (a) M. S. Bartlett and J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 545 (1949); (b) T. F. Jordan and E. C. G. Sudarshan, Rev. Mod. Phys. **33**, 515 (1961); (c) E. C. G. Sudarshan, *Lectures in Theoretical Physics* (W. A. Benjamin Company, Inc., New York, 1962), Vol. 2, p. 178.

[19] C. L. Mehta, J. Math. Phys. **5**, 677 (1964).

III. TIME EVOLUTION OF COHERENT STATES —EXPLICIT SOLUTION

According to Eq. (2.14) the general form of the Hamiltonian (in the Heisenberg picture) consistent with the requirement that the states which are initially coherent remain coherent at all times is given by

$$\hat{H}=\sum_\lambda\sum_\mu \omega_{\lambda\mu}(t)\hat{a}_\lambda^\dagger(t)\hat{a}_\mu(t) +\sum_\lambda\{F_\lambda(t)\hat{a}_\lambda^\dagger(t)+F_\lambda^*(t)\hat{a}_\lambda(t)\}+\beta(t). \quad (3.1)$$

The operator $\hat{a}_\lambda$ therefore satisfies the following equation of motion:

$$\begin{aligned} d\hat{a}_\lambda/dt&=-i[\hat{a},\hat{H}] \\ &=-i\sum \omega_{\lambda\mu}(t)\hat{a}_\mu(t)-iF_\lambda(t). \end{aligned} \quad (3.2)$$

Let us rewrite Eq. (3.2) in a matrix notation

$$d\hat{a}/dt=-i\omega(t)\hat{a}(t)-iF(t), \quad (3.3)$$

where $\hat{a}$ and F are the column vectors

$$\hat{a}(t)=\begin{pmatrix}\hat{a}_1(t)\\ \vdots\\ \hat{a}_\lambda(t)\\ \vdots\end{pmatrix}, \quad F(t)=\begin{pmatrix}F_1(t)\\ \vdots\\ F_\lambda(t)\\ \vdots\end{pmatrix}, \quad (3.4)$$

and ω is the Hermitian matrix

$$\omega(t)=\begin{pmatrix}\omega_{11}(t)\cdots & \omega_{1\lambda}(t)\cdots\\ \vdots & \vdots\\ \omega_{\lambda1}(t)\cdots & \omega_{\lambda\lambda}(t)\cdots\\ \vdots & \vdots\end{pmatrix}. \quad (3.5)$$

In order to solve Eq. (3.3), let us first assume $F(t)=0$. In this case we have

$$d\hat{a}/dt=-i\omega(t)\hat{a}(t), \quad (3.6)$$

whose formal solution is given by[20]

$$\hat{a}(t)=U(t)\hat{a}(0). \quad (3.7)$$

Here $U(t)$ is the unitary matrix

$$U(t)=\left\{\exp\left[-i\int_0^t\omega(t')dt'\right]\right\}_+ \quad (3.8)$$

and the subscript $+$ denotes the time-ordering opera-

[20] See for example F. R. Gantemacher, *Applications of the Theory of Matrices* (Interscience Publishers, Inc., New York, 1959), Chap. IV.

tion defined by the following relation[21]:

$$\left\{\exp\left(-i\int_0^t \omega(t')dt'\right)\right\}_+ = 1+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\left\{\int_0^t dt_1\cdots\int_0^t dt_n\,\omega(t_1)\cdots\omega(t_n)\right\}_+ = 1+\sum_{n=1}^{\infty}(-i)^n\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n \times\omega(t_1)\omega(t_2)\cdots\omega(t_n). \quad (3.9)$$

To verify that (3.7) is the solution of (3.6), one only has to differentiate (3.7) and use the following relation which is obtained by differentiating (3.8) and using (3.9)

$$\dot{U}(t)=-i\omega(t)U(t). \quad (3.10)$$

In the case when the matrices $\omega(t)$ and $\omega(t')$ commute, i.e., when

$$[\omega(t),\omega(t')]\equiv 0,\quad \text{for all } t, t', \quad (3.11)$$

we can write

$$U(t)=\left\{\exp\left(-i\int_0^t \omega(t')dt'\right)\right\}_+ =\exp\left(-i\int_0^t \omega(t')dt'\right). \quad (3.12)$$

However in the general case when $[\omega(t),\omega(t')]\neq 0$, $U(t)$ cannot be expressed in such a closed form. On the other hand if we are given the unitary operator $U(t)$, we can readily evaluate $\omega(t)$ in all cases from the relation

$$\omega(t)=i\dot{U}(t)U^\dagger(t)=-iU(t)\dot{U}^\dagger(t) =\tfrac{1}{2}i[\dot{U}(t)U^\dagger(t)-U(t)\dot{U}^\dagger(t)], \quad (3.13)$$

which is obtained on multiplying Eq. (3.10) by $iU^\dagger(t)$ on the right and using the fact that $U(t)$ is unitary.

One can now write the solution of (3.3) in the more general case when $F(t)\neq 0$

$$\hat{a}(t)=U(t)\hat{a}(0)-iU(t)\int_0^t U^\dagger(t')F(t')dt', \quad (3.14)$$

[21] Such time-ordering operations are also used in quantum field theory; see for example S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961), pp. 330–334.

Alternatively one can write $U(t)$ as a "product integral" in the form

$$U(t)=\left\{\exp\left(-i\int_0^t \omega(t')dt'\right)\right\}_+ =\lim_{\substack{N\to\infty\\ N\delta t=t}}\prod_{n=0}^{N}\exp\{-i\delta t\omega(t-n\delta t)\}.$$

which, when written in the explicit form, reads

$$\hat{a}_\lambda(t)=\sum_\mu U_{\lambda\mu}(t)\hat{a}_\mu(0) -i\sum_\mu\sum_\sigma U_{\lambda\mu}(t)\int_0^t U_{\mu\sigma}{}^\dagger(t')F_\sigma(t')dt'. \quad (3.15)$$

Equation (3.14) satisfies the boundary condition at $t=0$ and the fact that it satisfies (3.3) can be verified by direct differentiation and making use of Eq. (3.10).

From (3.15) we see that the eigenvalue of the operator $\hat{a}_\lambda(t)$ is given by

$$v_\lambda(t)=\sum_\mu U_{\lambda\mu}(t)v_\mu(0) -i\sum_\mu\sum_\sigma U_{\lambda\mu}(t)\int_0^t U_{\mu\sigma}{}^\dagger(t')F_\sigma(t')dt'. \quad (3.16)$$

Explicit solution can also be obtained in the Schrödinger picture. In this case the operator $\hat{a}_\lambda$ is time-independent; however, its expectation value does depend on time and satisfies the equation of motion

$$i\frac{d}{dt}\langle\{v(t)\}|\hat{a}_\lambda|\{v(t)\}\rangle=\langle\{v(t)\}|[\hat{a},\hat{H}]|\{v(t)\}\rangle. \quad (3.17)$$

If $|\{v(t)\}\rangle$ is an eigenstate of $\hat{a}_\lambda$, we see from (2.20) that $\hat{H}$ must be of the form

$$\hat{H}=\sum_\lambda\sum_\mu \omega_{\lambda\mu}(t)\hat{a}_\lambda{}^\dagger\hat{a}_\mu +\sum_\lambda\{F_\lambda(t)\hat{a}_\lambda{}^\dagger+F_\lambda{}^*(t)\hat{a}_\lambda\}+\beta(t). \quad (3.18)$$

Equation (3.17) then gives

$$\dot{v}_\lambda(t)=-i\sum_\mu \omega_{\lambda\mu}(t)v_\mu(t)-iF_\lambda(t), \quad (3.19)$$

an equation similar to (3.2). Hence if we proceed in a similar manner as in connection with (3.2) we obtain

$$v_\lambda(t)=\sum_\mu U_{\lambda\mu}(t)v_\mu(0) -i\sum_\mu\sum_\sigma U_{\lambda\mu}(t)\int_0^t U_{\mu\sigma}{}^\dagger(t')F_\sigma(t')dt', \quad (3.20)$$

where $U(t)$ is the unitary matrix given by (3.8).

We see that the eigenvalues obtained in both the Heisenberg and the Schrödinger pictures [Eqs. (3.16) and (3.20)] are identical.

The state $|\{v(t)\}\rangle$ which is an eigenstate of $\hat{a}_\lambda$ with eigenvalue $v_\lambda(t)$ (for all λ and for all t) is thus given by the following equation

$$|\{v(t)\}\rangle=\prod_\lambda|v_\lambda(t)\rangle =\exp\{\sum_\lambda(v_\lambda(t)\hat{a}_\lambda{}^\dagger-v_\lambda{}^*(t)a_\lambda)\}|0\rangle, \quad (3.21)$$

where $v_\lambda(t)$ is given by (3.20) and $|0\rangle$ denotes the vacuum state.

IV. FREQUENCY SPECTRUM OF THE ANNIHILATION OPERATORS

In Sec. III we determined the time dependence of the annihilation operators in presence of an interaction such that the states which are initially coherent do not change their essential character. We see from (3.15) that this time dependence is given by

$$\hat{a}_\lambda(t)=\sum_\mu U_{\lambda\mu}(t)\hat{a}_\mu(0)-i\sum_\mu\sum_\sigma U_{\lambda\mu}(t)\int_0^t U_{\mu\sigma}{}^\dagger(t')F_\sigma(t')dt', \quad (4.1)$$

where $U(t)$ is the unitary matrix

$$U_{\lambda\mu}(t)=\left\{\exp\left(-i\int_0^t\omega(t')dt'\right)\right\}_{+\lambda\mu} \quad (4.2)$$

and the subscript $+$ denotes time ordering defined by (3.9).

We now wish to study the frequency spectrum $\hat{b}_\lambda(\nu)$ of the annihilation operator[22] $\hat{a}_\lambda(t)$. By definition $\hat{b}_\lambda(\nu)$ and $\hat{a}_\lambda(t)$ form a Fourier-transform pair

$$\hat{b}_\lambda(\nu)=\int_{-\infty}^\infty \hat{a}_\lambda(t)e^{2\pi i\nu t}dt, \quad (4.3)$$

$$\hat{a}_\lambda(t)=\int_{-\infty}^\infty \hat{b}_\lambda(\nu)e^{-2\pi i\nu t}d\nu. \quad (4.4)$$

We will first consider the simple case of a one-dimensional harmonic oscillator.

A. Systems with One Degree of Freedom

In this case we have from (4.1), suppressing the mode-labeling index λ,

$$\hat{a}(t)=U(t)\hat{a}(0)-iU(t)\int_0^t U^*(t')F(t')dt', \quad (4.5)$$

where

$$U(t)=\exp\left(-i\int_0^t\omega(t')dt'\right). \quad (4.6)$$

The frequency spectrum $\hat{b}(\nu)$ is related to $\hat{a}(t)$ by the Fourier transform relations

$$\hat{b}(\nu)=\int_{-\infty}^\infty \hat{a}(t)e^{2\pi i\nu t}dt, \quad (4.7)$$

$$\hat{a}(t)=\int_{-\infty}^\infty \hat{b}(\nu)e^{-2\pi i\nu t}d\nu. \quad (4.8)$$

Let us discuss some special cases:

Case 1. Free Hamiltonian. In this case $\omega(t)\equiv\omega_0>0$ and $F(t)\equiv 0$. Equations (4.5) and (4.6) then give

$$\hat{a}(t)=e^{-i\omega_0 t}\hat{a}(0), \quad (4.9)$$

so that

$$\hat{b}(\nu)=\delta(\nu-\omega_0/2\pi)\hat{a}(0) \quad (4.10)$$

and we have a single positive frequency at $\nu=\omega_0/2\pi$. This means that we can refer to the time-dependent annihilation operator $\hat{a}(t)$ as the positive-frequency part of the canonical variable $\hat{q}=(\hat{a}+\hat{a}^\dagger)/(2\omega_0)^{1/2}$ and hence the operator $\hat{a}(t)$ is an "analytic signal."

Case 2. Forced oscillator. Let us next consider the case when $\omega(t)\equiv\omega_0>0$, but $F(t)\neq 0$. In this case Eq. (4.5) gives

$$\hat{a}(t)=e^{-i\omega_0 t}\hat{a}(0)-ie^{-i\omega_0 t}\int_0^t e^{i\omega_0 t'}F(t')dt'. \quad (4.11)$$

Let the frequency spectrum of the forcing term $F(t)$ be given by $f(\nu)$, i.e., that $F(t)$ is given by

$$F(t)=\int_{-\infty}^\infty f(\nu)e^{-2\pi i\nu t}d\nu. \quad (4.12)$$

Equation (4.11) then gives

$$\hat{a}(t)=e^{-i\omega_0 t}\hat{a}(0)+\int_{-\infty}^\infty \frac{e^{-i\omega_0 t}-e^{-2\pi i\nu t}}{\omega_0-2\pi\nu}f(\nu)d\nu. \quad (4.13)$$

On taking the Fourier transform of (4.13), we obtain the following expression for $\hat{b}(\nu)$:

$$\hat{b}(\nu)=\delta\left(\nu-\frac{\omega_0}{2\pi}\right)\left\{\hat{a}(0)-\frac{1}{2\pi}\int_{-\infty}^\infty P\frac{f(\nu')d\nu'}{\nu'-\nu}\right\}-P\frac{f(\nu)}{\omega_0-2\pi\nu}, \quad (4.14)$$

where P denotes the principal-value function.

Equation (4.14) shows that depending on whether $f(\nu)$ does or does not vanish for negative frequencies, $\hat{b}(\nu)$ also does or does not vanish for negative frequencies. This means that the annihilation operator $\hat{a}(t)$ is an analytic signal if and only if $F(t)$ is an analytic signal. If $F(t)$ is an analytic signal, it must be a complex-valued function, i.e., that the real or the imaginary part of F is not identically zero:

$$F+F^*\neq 0;\quad F-F^*\neq 0. \quad (4.15)$$

[22] In this section we work in Heisenberg picture so that the operator $\hat{a}_\lambda$ is explicitly time-dependent. In Schrödinger picture, by frequency spectrum of the annihilation operator $\hat{a}_\lambda$ we would mean the Fourier transform of its eigenvalue

$$b_\lambda(\nu)=\int_{-\infty}^\infty v_\lambda(t)e^{2\pi i\nu t}dt,$$

so that if one wants to work in Schrödinger picture, one would replace the operator $\hat{b}(\nu)$ of the Heisenberg picture by its eigenvalue $b(\nu)$.

For, under certain general conditions, the real and imaginary parts of $F(t)$ are related by Hilbert transform relations[23,10] and hence the identically vanishing of either implies the identically vanishing of the other. Now the term containing $F(t)$ in the interacting Hamiltonian can be written as [cf. Eq. (2.14)]

$$F(t)\hat{a}^\dagger(t)+F^*(t)\hat{a}(t) =\frac{F+F^*}{2}(\hat{a}+\hat{a}^\dagger)-\frac{F-F^*}{2}(\hat{a}-\hat{a}^\dagger). \quad (4.16)$$

Hence if F satisfies (4.15), the interaction Hamiltonian contains terms both proportional to the coordinate $[(\hat{a}+\hat{a}^\dagger)/(2\omega)^{1/2}]$ as well as the momentum $[-i\sqrt{\omega}\times(\hat{a}-\hat{a}^\dagger)/\sqrt{2}]$. We therefore conclude the following: In order that $\hat{a}(t)$ be an analytic signal, it is necessary that the interaction contains coordinate-dependent as well as velocity-dependent potentials.

For the usual velocity-independent interactions, $F(t)$ is real so that (4.15) is not satisfied and hence $F(t)$ is not an analytic signal. For such cases of course, $\hat{a}(t)$ is also not an analytic signal.

Case 3. Oscillator with time-dependent frequency and $F(t)\equiv 0$. For this case we have, from Eqs. (4.5) and (4.6),

$$\hat{a}(t)=\exp\left(-i\int_0^t \omega(t')dt'\right)\hat{a}(0), \quad (4.17)$$

so that[24]

$$\hat{b}(\nu)=\int_{-\infty}^{\infty}\exp\left(-i\int_0^t\omega(t')dt'\right)e^{2\pi i\nu t}dt\;\hat{a}(0). \quad (4.18)$$

It can be seen that, in general, both positive- and negative-frequency components exist.

If, however, we require that only positive-frequency components should occur then $\exp(-i\int_0^t\omega(t')dt')$ is an analytic signal. Further since $\omega(t)$ is real, this function is unimodular. The most general form of the unimodular analytic signal $\exp(-i\int_0^t\omega(t')dt')$ is then given by[25]

$$\exp\left(-i\int_0^t\omega(t')dt'\right) =\exp(-i\gamma-i\omega_0 t)\prod_k\frac{t-\alpha^{(k)}+i\beta^{(k)}}{t-\alpha^{(k)}-i\beta^{(k)}}, \quad (4.19)$$

where $\omega_0\geq 0$, $\beta^{(k)}\geq 0$; γ and $\alpha^{(k)}$ are real constants and $\prod_k$ denotes product over an arbitrary number of factors. The constant γ is so chosen that the boundary condition at $t=0$ is satisfied. From (4.19), we have

$$\int_0^t\omega(t')dt' =\omega_0 t-2\sum_k\left\{\tan^{-1}\left(\frac{\beta^{(k)}}{t-\alpha^{(k)}}\right)+\tan^{-1}\left(\frac{\beta^{(k)}}{\alpha^{(k)}}\right)\right\}, \quad (4.20)$$

so that, on differentiation, we obtain

$$\omega(t)=\omega_0+2\sum_k\frac{\beta^{(k)}}{(t-\alpha^{(k)})^2+\beta^{(k)2}}. \quad (4.21)$$

Thus we see that when $F(t)=0$, the annihilation operator $a(t)$ is an analytic signal if and only if the time dependence of the frequency of the oscillator $\omega(t)$ is of the form (4.21).

To see an explicit example let us consider the case when only two terms are present under the summation sign in (4.21). Further, in order that the frequency spectrum $\hat{b}(\nu)$ turn out to be real, we make a special choice of these two terms such that $\omega(t)$ is given by

$$\omega(t)=\omega_0+\frac{2\beta}{(t-\alpha)^2+\beta^2}+\frac{2\beta}{(t+\alpha)^2+\beta^2}; \qquad \beta>0,\ \omega_0>0. \quad (4.22)$$

This behavior of $\omega(t)$ with respect to t is shown in Fig. 1. In this case we obtain from (4.18)

$$\begin{aligned}\hat{b}(\nu)&=\int_{-\infty}^{\infty}e^{i(2\pi\nu-\omega_0)t}\frac{(t-\alpha+i\beta)(t+\alpha+i\beta)}{(t-\alpha-i\beta)(t+\alpha-i\beta)}dt\;\hat{a}(0)\\ &=\int_{-\infty}^{\infty}e^{i(2\pi\nu-\omega_0)t}\left\{1+\frac{4it\beta}{(t-\alpha-i\beta)(t+\alpha-i\beta)}\right\}dt\;\hat{a}(0).\end{aligned} \quad (4.23)$$

If we carry out the integration on the right-hand side

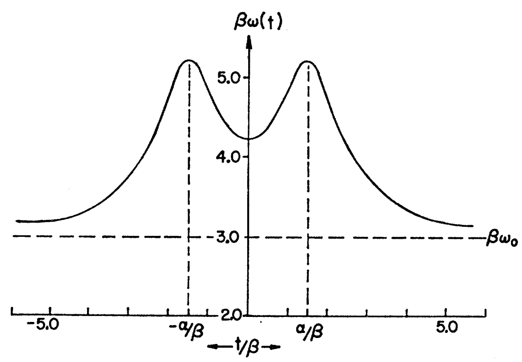

FIG. 1. Time dependence of the frequency of an interacting harmonic oscillator in a special case when $\hat{a}(t)$ is an analytic signal.

[23] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon Press, Oxford, England, 1948), 2nd ed., p. 128.

[24] The function $\exp(-i\int_0^t\omega(t')dt')$ is unimodular and hence is not square integrable. The Fourier transform of such functions is to be taken in the sense of generalized function theory.

[25] See for example S. F. Edwards and G. B. Parrent, Jr. [Opt. Acta **6**, 367 (1959)] where the most general form of unimodular analytic signal is derived. See also N. G. van Kampen, Phys. Rev. **89**, 1072 (1953).

of (4.23) we obtain

$$\hat{b}(\nu)=\hat{a}(0)\left\{\delta\left(\nu-\frac{\omega_0}{2\pi}\right)+\frac{8\pi\beta}{\alpha}(\alpha^2+\beta^2)^{1/2}e^{-\beta(2\pi\nu-\omega_0)}\sin\{\alpha(2\pi\nu-\omega_0)-\tan^{-1}(\alpha/\beta)\}\theta(2\pi\nu-\omega_0)\right\};\quad(4.24)$$

where $\theta(x)$ is the Heaviside step function

$$\begin{aligned}\theta(x)&=1\quad\text{if}\quad x>0,\\&=0\quad\text{if}\quad x<0.\end{aligned}\qquad(4.25)$$

The eigenvalue $b(\nu)$ of the operator $\hat{b}$ as given by (4.24) is plotted in Fig. 2.

Let us consider next an explicit case when $\omega(t)$ is not of the form (4.21) but is given by

$$\omega(t)=\omega_0-\frac{2\beta}{(t-\alpha)^2+\beta^2}-\frac{2\beta}{(t+\alpha)^2+\beta^2};\quad \omega_0>0,\beta>0.\quad(4.26)$$

If we substitute (4.26) in (4.6) and use Eqs. (4.7) and (4.5) we obtain, after simplification, the following expression for $\hat{b}(\nu)$:

$$\hat{b}(\nu)=\hat{a}(0)\left\{\delta\left(\nu-\frac{\omega_0}{2\pi}\right)+\frac{8\pi\beta}{\alpha}(\alpha^2+\beta^2)^{1/2}e^{-\beta(\omega_0-2\pi\nu)}\sin\{\alpha(\omega_0-2\pi\nu)-\tan^{-1}(\alpha/\beta)\}\theta(\omega_0-2\pi\nu)\right\}.\quad(4.27)$$

We see from (4.27) that $\hat{a}(t)$ has negative-frequency components also and hence is no longer an analytic signal, as was expected. For comparison, we plot $\omega(t)$ given by (4.26) in Fig. 3 and the eigenvalue $b(\nu)$ of the operator $\hat{b}$ which is given by (4.27) in Fig. 4.

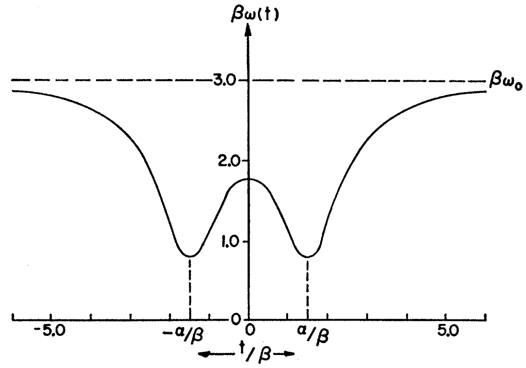

FIG. 3. Time dependence of the frequency of an interacting harmonic oscillator in a special case when $\hat{a}(t)$ is not an analytic signal.

Case 4. Forced oscillator with time-dependent frequency. In the general case when the frequency ω is time dependent and the forcing term $F(t)\neq 0$, we have [Eq. (4.5)]

$$\hat{a}(t)=U(t)\hat{a}(0)-iU(t)\int_0^t U^*(t')F(t')dt',\quad(4.28)$$

where $U(t)$ is given by (4.6). If we denote the Fourier transforms of $U(t)$ and $F(t)$ by $u(\nu)$ and $f(\nu)$ respectively, i.e., if

$$u(\nu)=\int_{-\infty}^{\infty}U(t)e^{2\pi i\nu t}dt,\quad(4.29)$$

$$f(\nu)=\int_{-\infty}^{\infty}F(t)e^{2\pi i\nu t}dt,\quad(4.30)$$

we obtain by taking the Fourier transform of (4.28),

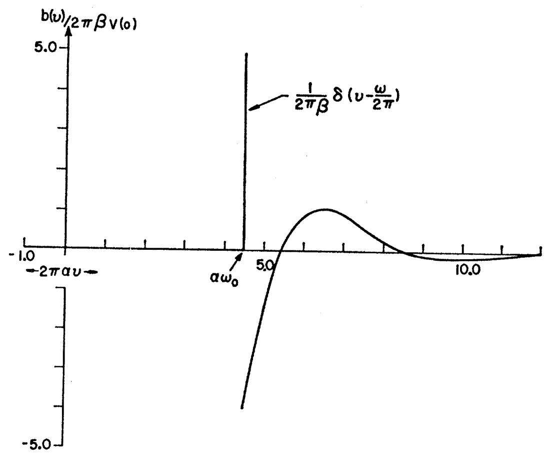

FIG. 2. Frequency spectrum of $\hat{a}(t)$ for an interacting harmonic oscillator with $\omega(t)$ given as in Fig. 1.

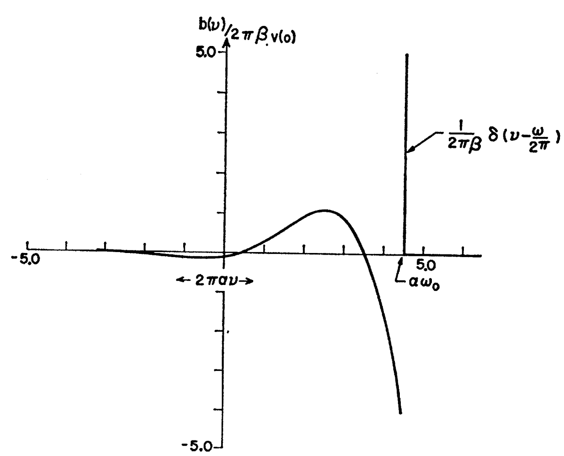

FIG. 4. Frequency spectrum of $\hat{a}(t)$ for an interacting harmonic oscillator with $\omega(t)$ given as in Fig. 3.

the following expression for $\hat{b}(\nu)$:

$$\hat{b}(\nu)=u(\nu)\hat{a}(0)+\int\int_{-\infty}^{\infty}\frac{u(\nu)-u(\nu+\nu')}{\nu'}\times u^*(\nu'')f(\nu''-\nu')d\nu'd\nu''. \quad (4.31)$$

We see from (4.31) that $\hat{a}(t)$ has positive- as well as negative-frequency components in its Fourier representation. Further, it can be seen that both the q-number part and the c-number part of $\hat{a}(t)$ [the first and second terms, respectively, on the right-hand side of (4.28)] may contain positive- as well as negative-frequency components.

It is also evident that if $\hat{a}(t)$ is an analytic signal (i.e., if it has only positive-frequency components), both the q-number part and the c-number parts must separately be analytic signals, since a negative-frequency contribution from one of them cannot cancel that from the other. We have seen in case 3 above that the q-number part, namely the term $U(t)\hat{a}(0)$, is an analytic signal if and only if $\omega(t)$ has the form given by (4.21), viz.,

$$\omega(t)=\omega_0+2\sum_k\frac{\beta^{(k)}}{(t-\alpha^{(k)})^2+\beta^{(k)2}}, \quad (4.32)$$

where ω_0 and $\beta^{(k)}$ are non-negative and $\alpha^{(k)}$ are real constants. If we denote the c-number part of $\hat{a}(t)$, by $A(t)$, i.e., if we write

$$A(t)=-iU(t)\int_0^t U^*(t')F(t')dt', \quad (4.33)$$

we have, using also (4.6),

$$F(t)=iU(t)\frac{d}{dt}\{U^*(t)A(t)\}$$
$$=i\dot{A}(t)-\omega(t)A(t). \quad (4.34)$$

Hence we conclude that $\hat{a}(t)$ is an analytic signal if and only if $\omega(t)$ is given by (4.32) and $F(t)$ is given by

$$F(t)=i\dot{A}(t)-\left(\omega_0+2\sum_k\frac{\beta^{(k)}}{(t-\alpha^{(k)})^2+\beta^{(k)2}}\right)A(t), \quad (4.35)$$

where $A(t)$ is an arbitrary analytic signal, with the only restriction $A(0)=0$.

We will now consider briefly systems with several degrees of freedom.

B. Systems with Several Degrees of Freedom

Let us rewrite Eqs. (4.1), (4.3), and (4.4) in matrix notation

$$\hat{a}(t)=U(t)\hat{a}(0)-iU(t)\int_0^t U^\dagger(t')F(t')dt', \quad (4.36)$$

$$\hat{b}(\nu)=\int_{-\infty}^{\infty}\hat{a}(t)e^{2\pi i\nu t}dt, \quad (4.37)$$

$$\hat{a}(t)=\int_{-\infty}^{\infty}\hat{b}(\nu)e^{-2\pi i\nu t}d\nu, \quad (4.38)$$

where $\hat{a}(t)$ and $F(t)$, are the column vectors given by (3.4), $U(t)$ is the unitary matrix given by (3.8) and $\hat{b}(\nu)$ is the column vector

$$\hat{b}(\nu)=\begin{pmatrix}\hat{b}_1(\nu)\\ \vdots\\ \hat{b}_\lambda(\nu)\\ \vdots\end{pmatrix}. \quad (4.39)$$

Further let us denote the Fourier transforms of $F(t)$ and $U(t)$ by $f(\nu)$ and $u(\nu)$, respectively, i.e.,

$$f(\nu)=\int_{-\infty}^{\infty}F(t)e^{2\pi i\nu t}dt, \quad (4.40)$$

$$u(\nu)=\int_{-\infty}^{\infty}U(t)e^{2\pi i\nu t}dt, \quad (4.41)$$

$$F(t)=\int_{-\infty}^{\infty}f(\nu)e^{-2\pi i\nu t}d\nu, \quad (4.42)$$

$$U(t)=\int_{-\infty}^{\infty}u(\nu)e^{-2\pi i\nu t}d\nu. \quad (4.43)$$

Substituting from (4.38), (4.42), (4.43) and its Hermitian adjoint in (4.36) and taking the Fourier transform of the resulting equation, we obtain after some simplifications the following expression for $\hat{b}(\nu)$:

$$\hat{b}(\nu)=u(\nu)\hat{a}(0)+\int\int_{-\infty}^{\infty}\frac{u(\nu)-u(\nu+\nu')}{\nu'}\times u^\dagger(\nu'')f(\nu''-\nu')d\nu'd\nu'', \quad (4.44)$$

i.e.,

$$b_\lambda(\nu)=\sum_\mu u_{\lambda\mu}(\nu)\hat{a}_\mu(0)+\int\int_{-\infty}^{\infty}\sum_\mu\sum_\sigma\frac{u_{\lambda\mu}(\nu)-u_{\lambda\mu}(\nu+\nu')}{\nu'}\times u_{\sigma\mu}{}^*(\nu'')f_\sigma(\nu''-\nu')d\nu'd\nu''. \quad (4.45)$$

We see again from (4.45) that $\hat{a}(t)$ has, in general, both positive and negative frequency components.

Let us denote the c-number part of $\hat{a}_\lambda(t)$ by $A_\lambda(t)$, i.e., let us write

$$A(t)=-iU(t)\int_0^t U^\dagger(t')F(t')dt', \quad (4.46)$$

where $A(t)$ is the column vector whose elements are $A_\lambda(t)$. On differentiating (4.46) with respect to t and rearranging terms, we then obtain

$$F(t)=i\dot{A}(t)-i\dot{U}(t)U^\dagger(t)A(t). \tag{4.47}$$

Now, if we require that $\hat{a}_\lambda(t)$ is an analytic signal, both the terms on the right-hand side of (4.36) must separately be analytic signals. This will be so if and only if $U(t)$ and $F(t)$ satisfy the following requirements:

(1) The matrix $U(t)$ is an analytic signal,[26] i.e.,

$$u(\nu)=0 \quad \text{for} \quad \nu<0; \tag{4.48}$$

(2) $F(t)$ is given by (4.47) where $A(t)$ is the column vector whose elements are arbitrary analytic signals subject to the condition $A(0)=0$.

The general form of a unitary matrix which is an analytic signal is not known. If, however, the unitary matrix $U(t)$ also satisfies the property that the commutator

$$[U(t),U(t')]=0, \quad \text{for all } t, t', \tag{4.49}$$

it is shown in the Appendix that $U(t)$ must be of the form

$$U(t)=V^\dagger e^{-i\gamma-i\omega_0 t}\prod_k B^{(k)}(t)V. \tag{4.50}$$

Here V is a unitary matrix, γ is a real diagonal matrix, ω_0 is a non-negative definite diagonal matrix and $\prod_k$ denotes product over an arbitrary number of Blaschke matrices

$$B^{(k)}(t)=(t\mathbf{1}-\alpha^{(k)}+i\beta^{(k)})(t\mathbf{1}-\alpha^{(k)}-i\beta^{(k)})^{-1}, \tag{4.51}$$

where $\mathbf{1}$ is the identity matrix, $\alpha^{(k)}$ are real diagonal matrices, and $\beta^{(k)}$ are non-negative definite diagonal matrices. All the matrices V, γ, ω_0, $\alpha^{(k)}$, and $\beta^{(k)}$ are time independent.

It is of interest to note that when $U(t)$ satisfies (4.49) or, equivalently, when $\omega(t)$ satisfies (3.11), we can write

$$U(t)=V^\dagger U_d(t)V, \tag{4.52}$$

where V is some time independent unitary matrix and $U_d(t)$ is a diagonal matrix. In this case, if we make a unitary transformation on the canonical annihilation operators $\hat{a}_\lambda(t)$, viz.,

$$\hat{a}_\lambda(t)\to a_\lambda'(t)=V_{\lambda\mu}\hat{a}_\mu(t), \tag{4.53}$$

we obtain from (4.1) the following expression for $\hat{a}_\lambda'(t)$:

$$\hat{a}_\lambda'(t)=\{U_d(t)\}_\lambda\hat{a}_\lambda'(0)-i\{U_d(t)\}_\lambda\int_0^t\{U_d^*(t')\}_\lambda \times\{VF(t')\}_\lambda dt'. \tag{4.54}$$

We note that unlike in (4.1), the different modes are now uncoupled and the problem becomes similar to that relating to systems with one degree of freedom.

APPENDIX: GENERAL FORM OF A UNITARY MATRIX THAT IS AN ANALYTIC SIGNAL

In this Appendix we will show that if a unitary matrix $U(t)$ is an analytic signal and satisfies the condition

$$[U(t),U(t')]=0 \quad \text{for all } t, t', \tag{A1}$$

it must be of the form given by (4.50).

Since $U(t)$ commutes with $U(t')$, we can find a time-independent unitary matrix V such that

$$U(t)=V^\dagger U_d(t)V, \tag{A2}$$

where $U_d(t)$ is a diagonal matrix whose elements are unimodular. Thus if $U(t)$ is an analytic signal, so is $U_d(t)$. The matrix elements of $U_d(t)$, which are now unimodular analytic signals, must therefore be of the form[25]

$$[U_d(t)]_\lambda=\exp(-i\gamma_\lambda-i\omega_{0\lambda}t)\prod_k B_\lambda^{(k)}. \tag{A3}$$

Here γ_λ is a real constant, $\omega_{0\lambda}$ is a non-negative constant and $\prod_k$ denotes product over an arbitrary number of Blaschke factors

$$B_\lambda^{(k)}=\frac{t-\alpha_\lambda^{(k)}+i\beta_\lambda^{(k)}}{t-\alpha_\lambda^{(k)}-i\beta_\lambda^{(k)}}, \tag{A4}$$

where $\alpha_\lambda^{(k)}$ are real and $\beta_\lambda^{(k)}$ are non-negative constants.

We can therefore write

$$U_d(t)=\exp(-i\gamma-i\omega_0 t)\prod_k(t\mathbf{1}-\alpha^{(k)}+i\beta^{(k)}) \times(t\mathbf{1}-\alpha^{(k)}-i\beta^{(k)})^{-1}, \tag{A5}$$

where γ and $\alpha^{(k)}$ are real diagonal matrices; ω_0 and $\beta^{(k)}$ are non-negative definite diagonal matrices, $\mathbf{1}$ is the identity matrix and $\prod_k$ denotes product over an arbitrary number of factors.

From (A5) and (A2), we conclude that $U(t)$ must be of the form given by (4.50), viz.,

$$U(t)=V^\dagger e^{-i\gamma-i\omega_0 t}\prod_k(t\mathbf{1}-\alpha^{(k)}+i\beta^{(k)}) \times(t\mathbf{1}-\alpha^{(k)}-i\beta^{(k)})^{-1}V. \tag{A6}$$

[26] A matrix will be said to be an analytic signal, if all its matrix elements are analytic signals.