Generalized Ward-Takahashi Identities and Current Algebras*

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Generalized Ward-Takahashi identities are written for current algebras generated by conserved and partially conserved currents. The application to meson-baryon scattering is discussed and expressions for the scattering lengths are derived. An approximation is obtained for the real part of the low-energy forward meson-nucleon scattering amplitude; this gives sum rules for the low-energy phase shifts. Some questions arising from off-mass-shell extrapolations are discussed.

I. INTRODUCTION

THE relations between measurable physical quantities, following from postulated equal-time commutation relations between vector and axial-vector currents and local-field or source operators, have attracted much interest during the past year. Fubini, Furlan, and Rossetti[1] showed that equal-time commutation relations of the kind suggested by Gell-Mann[2] lead to a large class of sum rules; in recent months several such relations have been obtained by various authors for a variety of particle phenomena.

In this paper we write down the generalized Ward-Takahashi identities for a current algebra generated by conserved and partially conserved currents, and use these for obtaining some results for two-body reactions.[3]

In addition to the simplest relations obtained by generalizing the Ward-Takahashi identities of electrodynamics,[4] we also write down the relations obtained from partially time-ordered operator products. These two sets of relations give a class of exact consequences of the current algebra (and of the conditions of conservation or partial conservation of the currents) for physical amplitudes and their absorptive parts.[5]

These exact consequences of the current algebra, which give information about amplitudes in certain unphysical limits, may be combined with dynamical assumptions like dispersion relations and unitarity to give simple low-energy approximations for physical amplitudes.

In Sec. II we write down the generalized Ward-Takahashi identities for the algebra generated by the time components of the vector and axial vector current densities. In Sec. III we derive a few relations for meson-baryon scattering; some questions arising in connection with off-mass-shell extrapolations are discussed in an Appendix. In Sec. IV we summarize our conclusions.

II. GENERALIZED WARD-TAKAHASHI IDENTITIES FOR THE ALGEBRA OF VECTOR AND AXIAL-VECTOR CURRENTS

In this section we shall write down the generalized Ward-Takahashi identities (GWTI) for the algebra generated by the time components $\mathcal{V}_\alpha{}^0(x)$ and $\mathcal{A}_\alpha{}^0(x)$ of the vector and axial-vector current densities. Methods similar to that used in obtaining the relations given here have been used by other authors.[6-8] However, we work throughout with vacuum expectation values of products of operators, and our relations differ in detail from those of the other authors; we shall therefore write them down here.

We first consider a conserved vector current. For simplicity, we shall write the equations for a four-point function; these may be immediately generalized to an n-point function. Let $\mathcal{V}_\alpha{}^\mu(x)$ denote a conserved vector current, $\varphi_\beta(y)$ a pseudoscalar field and $\psi_\gamma(z)$ a spin-$\frac{1}{2}$ baryon field. The Greek subscripts α, β, γ, $\cdots$ refer to the SU_3 component; for definiteness we consider SU_3 octets here. Current conservation gives

$$\partial_\mu \mathcal{V}_\alpha{}^\mu(x)=0. \tag{2.1}$$

Define

$$\mathcal{T}_V{}^\mu=\int d^4x d^4y d^4z d^4w \times[\exp i(p_f\cdot z-p_i\cdot w+q_2\cdot y-q_1\cdot x)]\tau_V{}^\mu, \tag{2.2}$$

where

$$\tau_V{}^\mu=\langle 0|T(\mathcal{V}_\alpha{}^\mu(x)\varphi_\beta(y)\psi_\gamma(z)\bar{\psi}_\sigma(w))|0\rangle. \tag{2.3}$$

The function (2.2) may be related to the transition matrix element for the process $V_\alpha+B_\sigma\rightarrow P_\beta+B_\gamma$, where B_γ, B_σ are spin-$\frac{1}{2}$ baryons, V_α is a vector meson, and P_β a pseudoscalar meson.

Using (2.1) and the commutation relations (CR's)

$$[\mathcal{V}_\alpha{}^0(x),\chi_\beta(y)]\delta(x_0-y_0)=if_{\alpha\beta\gamma}\chi_\gamma(y)\delta(x-y), \tag{2.4}$$

where

$$\chi_\beta(y)=\psi_\beta(y) \quad\text{or}\quad \varphi_\beta(y),$$

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[1] S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40**, 1174 (1965), and various subsequent papers.

[2] M. Gell-Mann, Physics **1**, 63 (1964); Phys. Rev. **125**, 1067 (1962).

[3] A brief summary of the basic ideas has been given by K. Raman and E. C. G. Sudarshan, Phys. Letters **21**, 450 (1966).

[4] J. C. Ward, Phys. Rev. **78**, 182 (1950); Y. Takahashi, Nuovo Cimento **6**, 371 (1957); T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[5] In electrodynamics the idea that the Ward-Takahashi identities could be used for expressing the restrictions arising from gauge invariance in a theory without a Lagrangian was suggested by Nishijima; see K. Nishijima, Phys. Rev. **119**, 485 (1960).

[6] V. Alessandrini, M. A. B. Bég, and L. S. Brown, Phys. Rev. **144**, 1137 (1966).

[7] S. Fubini, Nuovo Cimento **43**, 475 (1966).

[8] W. Weisberger, Phys. Rev. **143**, 1302 (1966).

we obtain

$$\frac{\partial}{\partial x^\mu}\tau_V{}^\mu(xyzw)=\delta(x-y)if_{\alpha\beta\beta'}\langle 0|T(\varphi_{\beta'}(y)\psi_\gamma(z)\bar{\psi}_\sigma(w))|0\rangle+\delta(x-z)if_{\alpha\gamma\gamma'}\langle 0|T(\varphi_\beta(y)\psi_{\gamma'}(z)\bar{\psi}_\sigma(w))|0\rangle$$
$$-\delta(x-w)if_{\alpha\sigma\sigma'}\langle 0|T(\varphi_\beta(y)\psi_\gamma(z)\bar{\psi}_{\sigma'}(w))|0\rangle. \quad (2.5)$$

Here, $if_{\alpha\beta\gamma}$ are the totally antisymmetric structure constants of SU_3.

Equation (2.5) gives the GWTI for the function (2.3) in configuration space; it is a direct generalization of the analogous result in electrodynamics. The Fourier transform of (2.5) gives the GWTI in momentum space

$$q_{1\mu}\mathcal{T}_V{}^\mu=f_{\alpha\beta\beta'}\mathfrak{F}_1(\beta'\gamma\sigma)+f_{\alpha\gamma\gamma'}\mathfrak{F}_2(\beta\gamma'\sigma)-f_{\alpha\sigma\sigma'}\mathfrak{F}_3(\beta\gamma\sigma'), \quad (2.6)$$

where $\mathcal{T}_V{}^\mu$ is defined by (2.2) and the functions $\mathfrak{F}_i$ are defined as follows:

$$\mathfrak{F}_1(\beta'\gamma\sigma)=\int d^4y d^4z d^4w\{\exp i[(q_2-q_1)\cdot y+p_f\cdot z-p_i\cdot w]\}\tau_1(y,z,w;\beta'\gamma\sigma), \quad (2.7)$$

$$\mathfrak{F}_2(\beta\gamma'\sigma)=\int d^4y d^4z d^4w\{\exp i[q_2\cdot y+(p_f-q_1)\cdot z-p_i\cdot w]\}\tau_2(y,z,w;\beta\gamma'\sigma), \quad (2.8)$$

$$\mathfrak{F}_3(\beta\gamma\sigma')=\int d^4y d^4z d^4w\{\exp i[q_2\cdot y+p_f\cdot z-(p_i+q_1)\cdot w]\}\tau_3(y,z,w;\beta\gamma\sigma'). \quad (2.9)$$

The functions τ_1, τ_2, and τ_3 are given by the vacuum expectation values on the right-hand side of (2.5) taken in that order.

We next write down the GWTI for a partially conserved axial-vector current satisfying

$$\partial_\mu\mathcal{A}_\alpha{}^\mu(x)=C_\alpha\varphi_\alpha(x). \quad (2.10)$$

Consider the functions $\tau_A{}^\mu$ and $\mathcal{T}_A{}^\mu$ obtained by replacing $\mathcal{V}_\alpha{}^\mu$ in (2.2) and (2.3) by $\mathcal{A}_\alpha{}^\mu$. Proceeding as before, we obtain instead of (2.6) the equation

$$q_{1\mu}\mathcal{T}_A{}^\mu=-i(\mu_\alpha{}^2-q_1{}^2)^{-1}C_\alpha\mathcal{T}_P+d_{\alpha\beta\beta'}\mathcal{H}_1(\beta'\gamma\sigma)+h_{\alpha\gamma\gamma'}\gamma_5\mathcal{G}_2(\beta\gamma'\sigma)-h_{\alpha\sigma\sigma'}\mathcal{G}_3(\beta\gamma\sigma')\gamma_5. \quad (2.11)$$

In obtaining this, we have assumed the CR's

$$[\mathcal{A}_\alpha{}^0(x),\psi_\beta(y)]\delta(x_0-y_0)=h_{\alpha\beta\gamma}i\gamma_5\psi_\gamma(y)\delta(x-y), \quad (2.12)$$

$$[\mathcal{A}_\alpha{}^0(x),\varphi_\beta(y)]\delta(x_0-y_0)=id_{\alpha\beta\gamma}\zeta_\gamma(y)\delta(x-y), \quad (2.13)$$

where $\zeta_\gamma(y)$ is a scalar operator. The coefficient $h_{\alpha\beta\gamma}$ may be obtained either by assigning the baryon field operator to a definite representation (or mixture of representations) of the algebra or by defining it in a composite-particle model. The coefficient $d_{\alpha\beta\gamma}$ in (2.13) was suggested by a quark model.

In (2.11), $\mathcal{T}_P$ and $\mathcal{H}_1$ are defined by

$$\mathcal{T}_P=\int d^4x d^4y d^4z d^4w \exp[i(p_f\cdot z-p_i\cdot w+q_2\cdot y-q_1\cdot x)](\mu_\alpha{}^2+\Box_x{}^2)\langle 0|T(\varphi_\alpha(x)\varphi_\beta(y)\psi_\gamma(z)\bar{\psi}_\sigma(w))|0\rangle; \quad (2.14)$$

and

$$\mathcal{H}_1(\beta'\gamma\sigma)=\int d^4y d^4z d^4w \exp[i(q_2-q_1)\cdot y+p_f\cdot z-p_i\cdot w]\langle 0|T(\zeta_{\beta'}(y)\psi_\gamma(z)\bar{\psi}_\sigma(w))|0\rangle. \quad (2.15)$$

Here $\zeta_{\beta'}(y)$ is defined by (2.13), and $\mathcal{G}_2$ and $\mathcal{G}_3$ are defined by equations analogous to (2.8) and (2.9).

The equations (2.6) and (2.11) may be generalized immediately to arbitrary n-point functions; these may be used for obtaining relations for production amplitudes. Such applications are under investigation.

We note that Eqs. (2.6) and (2.11) are exact relations valid for all momenta, and follow directly from the current algebra without any use of the analytic properties of the amplitudes. By combining these relations with assumptions about the analytic properties and other dynamical assumptions like unitarity, one may derive various relations for physical amplitudes.

Further restrictions implied by the current algebra are obtained by adding a second set of equations, obtained by replacing the T product in Eq. (2.3), for instance, by a partially time-ordered product:

$$\tau'_V{}^\mu(yzw;x)=\langle 0|T(\varphi_\beta(y)\psi_\gamma(z)\bar{\psi}_\sigma(w))\mathcal{V}_\alpha{}^\mu(x)|0\rangle. \quad (2.16)$$

We define $\mathcal{T}'_V{}^\mu$ as the Fourier transform, defined as in (2.2), of $\tau'_V{}^\mu$.

Going through the same procedure as above, we obtain the following equations:

$$q_{1\mu}\mathcal{T}'_V{}^\mu=0, \quad (2.17)$$

$$q_{1\mu}\mathcal{T}'_A{}^\mu=(\mu_\alpha{}^2-q_1{}^2)^{-1}\mathcal{T}'_P, \quad (2.18)$$

where $\mathcal{T}'_A{}^\mu$ and $\mathcal{T}'_P$ are defined by taking $\mathcal{A}_\alpha{}^\mu$ and φ_α outside the time-ordering in the definitions of $\mathcal{T}_A{}^\mu$ and $\mathcal{T}_P$.

Here, all quantities defined by replacing the T products by a partially ordered product as in (2.16) will be denoted by a prime. When we write these equations in terms of partially ordered R products, they give equations for the absorptive part of the amplitude.

Equations (2.6) and (2.17), and Eqs. (2.11) and (2.18), are the generalized Ward-Takahashi identities for the functions $\mathcal{T}_V{}^\mu$ and $\mathcal{T}_A{}^\mu$, respectively, and their absorptive parts. When the baryons are on the mass shell, Eqs. (2.6) and (2.11) reduce to equations derived by Alessandrini, Bég, and Brown,[6] and by Fubini,[7] and Eqs. (2.17) and (2.18) to equations derived by Fubini.[7]

The function $T_V{}^\mu$, defined by

$$\begin{aligned}i(2\pi)^4\delta(p_i+q_1-p_f-q_2)T_V{}^\mu &=\bar{u}(p_f)(-\gamma\cdot p_f+m_f)(\mu_\beta{}^2-q_2{}^2)\\ &\quad\times\mathcal{T}_V{}^\mu(-\gamma\cdot p_i+m_i)u(p_i), \quad (2.19)\end{aligned}$$

and the corresponding function $T_A{}^\mu$ describes matrix elements with all the external momenta off the mass shell.

In the special case where the external momenta are on the mass shell, Eqs. (2.6) and (2.17) give

$$q_{1\mu}T_V{}^\mu=0;\quad q_{1\mu}T'_V{}^\mu=0, \quad (2.20)$$

which are the conditions imposed by current conservation, while (2.11) and (2.18) give

$$q_{1\mu}T_A{}^\mu=\frac{-iC_\alpha}{\mu_\alpha{}^2-q_1{}^2}T_P,\quad q_{1\mu}T'_A{}^\mu=\frac{-iC_\alpha}{\mu_\alpha{}^2-q_1{}^2}T'_P, \quad (2.21)$$

which are generalized Goldberger-Treiman relations. The amplitude T_P is defined by replacing $T_V{}^\mu$ and $\mathcal{T}_V{}^\mu$ in (2.19) by T_p and $\mathcal{T}_P$, respectively, and using (2.14) for $\mathcal{T}_P$. $T'_V{}^\mu$ and $T'_A{}^\mu$ are defined in terms of $\mathcal{T}'_V{}^\mu$ and $\mathcal{T}'_A{}^\mu$ by equations similar to (2.19).

When one or more of the momenta q_2, p_i, or p_f are kept off the mass shell, additional terms appear in the expressions for $q_{1\mu}T_V{}^\mu$ and $q_{1\mu}T_A{}^\mu$. Thus when q_2 is kept off the mass shell and p_i, p_f on the mass shell, one obtains

$$q_{1\mu}T_V{}^\mu=f_{\alpha\beta\beta'}F_1, \quad (2.22)$$

where F_1 is defined by replacing $T_V{}^\mu$ and $\mathcal{T}_V{}^\mu$ in (2.19) by F_1 and $\mathfrak{F}_1$, respectively, and using the definition (2.7) of $\mathfrak{F}_1$. Equation (2.22) gives the consequence of current conservation for the "photoproduction" amplitude $T_V{}^\mu$ when the meson φ_β [See Eq. (2.3)] is off the mass shell.

If p_i or p_f is kept off the mass shell, there is an analogous contribution from the F_2 or F_3 term in (2.6). Similar statements can be made for $q_{1\mu}T_A{}^\mu$. Note that for $T'_V{}^\mu$ and $T'_A{}^\mu$, (2.20) and (2.21) still hold.

In using a relation such as (2.6) and (2.11) in the limit $q_1\to 0$, one should separate the proper and improper parts on either side of the equation; the ambiguities arising from the improper parts in the limit $q_1\to 0$ are then seen to cancel each other, as observed by Alessandrini, Bég, and Brown,[6] and by Weisberger.[8]

III. APPLICATION TO TWO-BODY REACTIONS; MESON-BARYON SCATTERING

We first write down the GWTI involving the amplitude for the scattering of a pseudoscalar meson on a spin-$\frac{1}{2}$ baryon.

Define the function

$$\begin{aligned}\tau_{AA}{}^{\mu\nu}(xyzw)\\ &=\langle 0|T(\mathcal{A}_{1\alpha}{}^\mu(x)\mathcal{A}_{2\beta}{}^\nu(y)\psi_\gamma(z)\bar{\psi}_\sigma(w))|\mathrm{p}\rangle, \quad (3.1)\end{aligned}$$

its Fourier transform

$$\begin{aligned}\mathcal{T}_{AA}{}^{\mu\nu}(q_1p_iq_2p_f)&=\int d^4xd^4yd^4zd^4w\\ &\quad\times[\exp i(p_f\cdot z-p_i\cdot w+q_2\cdot y-q_1\cdot x)]\tau_{AA}{}^{\mu\nu}, \quad (3.2)\end{aligned}$$

and the amplitudes $M_{AA}{}^{\mu\nu}$ and $T_{AA}{}^{\mu\nu}$:

$$\begin{aligned}i(2\pi)^4\delta(p_i+q_1-p_f-q_2)M_{AA}{}^{\mu\nu}\\ &=(-\gamma\cdot p_f+m_f)\mathcal{T}_{AA}{}^{\mu\nu}(-\gamma\cdot p_i+m_i); \quad (3.3)\end{aligned}$$

$$T_{AA}{}^{\mu\nu}=\bar{u}(p_f)M_{AA}{}^{\mu\nu}u(p_i). \quad (3.4)$$

We take all the operators here to be octet operators under SU_3.

When evaluating the divergence of (3.1), first with respect to x^μ and then with respect to y^ν, we encounter terms involving the equal-time commutators $[\mathcal{A}_\alpha{}^0(x),\mathcal{A}_\beta{}^\nu(y)]$ and $[\mathcal{A}_\beta{}^0(x),\varphi_\alpha(y)]$. For these we assume the following CR's:

$$\begin{aligned}[\mathcal{A}_\alpha{}^0(x),\mathcal{A}_\beta{}^\nu(y)]\delta(x_0-y_0)&=if_{\alpha\beta\gamma}\mathcal{V}_\gamma{}^\nu(y)\delta(x-y)\\ &\quad+(1-\delta^{\nu 0})\frac{\partial}{\partial x^\nu}[\rho_{\alpha\beta}(x)\delta(x-y)]. \quad (3.5)\end{aligned}$$

(Here ν is not summed over in the second term on the right.)

$$[\mathcal{A}_\beta{}^0(x),\varphi_\alpha(y)]\delta(x_0-y_0)=id_{\beta\alpha\alpha'}\zeta_{\alpha'}(y)\delta(x-y). \quad (3.6)$$

In (3.6), $\zeta_{\alpha'}(y)$ denotes a scalar operator; the coefficient $d_{\beta\alpha\alpha'}$ is suggested by a quark model. In (3.5), the second term on the right-hand side is the simplest form of the Schwinger term,[9] with $\rho_{\alpha\beta}(x)=\rho_{\beta\alpha}(x)$.

When p_i, p_f are kept on the mass shell and q_1, q_2 off the mass shell, we obtain the following relations:

$$q_{1\mu}T_{AA}{}^{\mu\nu}=-i(\mu_\alpha{}^2-q_1{}^2)^{-1}C_\alpha T_{PA}{}^\nu -if_{\alpha\beta\beta'}F_{\beta'}{}^\nu(\beta'\gamma\sigma)-[1-\delta^{\nu 0}]q_1{}^\nu S_{\alpha\beta}; \quad (3.7)$$

$$-q_{2\nu}T_{PA}{}^\nu=-i(\mu_\beta{}^2-q_2{}^2)^{-1}C_\beta T_{PP} -id_{\beta\alpha\alpha'}\mathcal{K}(\alpha'\gamma\sigma); \quad (3.8)$$

$$q_{1\mu}q_{2\nu}T_{AA}{}^{\mu\nu}=(\mu_\alpha{}^2-q_1{}^2)^{-1}(\mu_\beta{}^2-q_2{}^2)^{-1}C_\alpha C_\beta T_{PP} +d_{\beta\alpha\alpha'}(\mu_\alpha{}^2-q_1{}^2)^{-1}C_\alpha\mathcal{K}(\alpha'\gamma\sigma) -if_{\alpha\beta\beta'}q_{2\nu}F_{\beta'}{}^\nu(\beta'\gamma\sigma)-(\mathbf{q}_1\cdot\mathbf{q}_2)S_{\alpha\beta}. \quad (3.9)$$

Here $T_{PA}{}^\nu$ is defined by replacing $\mathcal{A}_{1\alpha}{}^\mu$ in (3.1) by the meson field operator $\varphi_{1\alpha}$, operating on the resulting function by $\mathcal{K}_x$, and substituting this for $\tau_{AA}{}^{\mu\nu}$ in (3.2)–(3.4). T_{PP} is defined by replacing $\mathcal{A}_{1\alpha}{}^\mu$ and $\mathcal{A}_{2\beta}{}^\nu$ by $\varphi_{1\alpha}$, $\varphi_{2\beta}$ respectively in (3.1), operating with $\mathcal{K}_x\mathcal{K}_y$ and substituting this for $\tau_{AA}{}^{\mu\nu}$ in (3.2)–(3.4). $\mathcal{K}_x$, $\mathcal{K}_y$ are the Klein-Gordon operators $(\mu_\alpha{}^2+\Box_x{}^2)$ and $(\mu_\beta{}^2+\Box_y{}^2)$, respectively. T_{PP} is the amplitude for the meson-baryon scattering process

$$\begin{array}{cccc} P_\alpha+B_\sigma & \rightarrow & P_\beta+B_\gamma, \\ q_1 \quad p_i & & q_2 \quad p_f \end{array} \quad (3.10)$$

with momenta as indicated; here, P_α and P_β are pseudoscalar mesons and B_σ and B_γ are spin-$\frac{1}{2}$ baryons; α, β, γ, σ are octet indices.

The function $S_{\alpha\beta}$ in (3.7) and (3.9) arises from the Schwinger term in (3.5), and is symmetric in α and β; in (3.7) there is no summation over ν.

$F_{\beta'}{}^\nu$ is defined by

$$F_{\beta'}{}^\nu(\beta'\gamma\sigma)=\int d^4x d^4z d^4w \exp i[(q_2-q_1)\cdot x+p_f\cdot z-p_i\cdot w]\bar{u}(p_f)(-\gamma\cdot p_f+m_f) \times\langle 0|T(\mathcal{V}_{\beta'}{}^\nu(x)\psi_\gamma(z)\bar{\psi}_\sigma(w))|0\rangle(-\gamma\cdot p_i+m)u(p_i), \quad (3.11)$$

and is related to the vector form factor of the baryon by

$$\langle p_f|\mathcal{V}_{\beta'}{}^\nu(0)|p_i\rangle=i^2\left[\frac{m_i m_f}{p_i{}^0 p_f{}^0}\right]^{1/2}F_{\beta'}{}^\nu. \quad (3.12)$$

As is well known, when the CR (3.5) contains the Schwinger terms the time-ordered function (3.1), and hence the amplitude (3.4), is not a Lorentz-covariant function.[10] By adding suitable terms to the time-ordered product one may obtain a covariant function and a covariant amplitude.[10] If $T_{AA}{}^{\mu\nu}$ in (3.7) and (3.9) is replaced by this covariant amplitude, we can obtain relations in which each term is covariant. In these relations, the last terms in (3.7) and (3.9) would be replaced by terms of the form $q_{1\mu}R_{\alpha\beta}{}^{\mu\nu}$ and $q_{1\mu}q_{2\nu}R_{\alpha\beta}{}^{\mu\nu}$, respectively, where $R_{\alpha\beta}{}^{\mu\nu}$ is a covariant, nonsingular function, with $R_{\alpha\beta}{}^{\mu\nu}=R_{\beta\alpha}{}^{\nu\mu}$; it may be zero in special cases.[11] A similar statement holds when the Schwinger term in (3.5) is of the more general form

$$\frac{\partial}{\partial x^k}[\rho_{\alpha\beta}{}^{kl}(x)\delta(x-y)], \quad k,l=1,2,3, \quad \text{with} \quad \rho_{\alpha\beta}{}^{kl}=\rho_{\beta\alpha}{}^{lk}, \quad \rho_{\alpha\beta}{}^{0k}=0=\rho_{\alpha\beta}{}^{k0}$$

(see Brown, Ref. 10). In the following we shall assume that we are dealing with Eqs. (3.7) and (3.9) written in terms of the covariant amplitudes; the latter will be written in the same notation $T_{AA}{}^{\mu\nu}$, etc.

The function $\mathcal{K}(\alpha'\gamma\sigma)$ is defined as

$$\mathcal{K}(\alpha'\gamma\sigma)=\int d^4x d^4z d^4w \exp i[(q_2-q_1)\cdot x+p_f\cdot z-p_i\cdot w](\mu_\alpha{}^2-q_1{}^2)\bar{u}(p_f)(-\gamma\cdot p_f+m_f) \times\langle 0|T(\zeta_{\alpha'}(x)\psi_\gamma(z)\bar{\psi}_\sigma(w))|0\rangle(-\gamma\cdot p_i+m_i)u(p_i). \quad (3.13)$$

The occurrence of a term of this type was noted by Weisberger[8]; the matrix element of the commutator

$$\left[\mathcal{A}_\alpha{}^0(x),\frac{\partial}{\partial y_0}\mathcal{A}_\beta{}^0(y)\right]\delta(x_0-y_0) \quad (3.14)$$

which is seen, on using the PCAC (partially conserved axial-vector current) hypothesis, to be closely related to the left-hand side of (3.6), has been studied by Kawarabayashi and Wada.[12] These authors show that in certain models this matrix element is small.

[9] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).

[10] K. Johnson, Nucl. Phys. **25**, 431 (1961); J. D. Bjorken, Phys. Rev. **148**, 1467 (1966); L. S. Brown, *ibid.* **150**, 1338 (1966); D. G. Boulware (to be published). We are grateful to Dr. D. G. Boulware and Dr. I. J. Muzinich for a discussion of the Schwinger terms.

[11] That $R_{\alpha\beta}{}^{\mu\nu}$ is a nonsingular function follows if one assumes that the covariant amplitude and the time-ordered amplitude have the same singularities in the finite part of the energy plane, and differ only in their asymptotic behavior.

[12] K. Kawarabayashi and W. Wada, Phys. Rev. **146**, 1209 (1966).

Since there is no evidence for the production of scalar mesons in reactions involving baryons, we expect that matrix elements like (3.13) are small.

For the absorptive parts $T'_{AA}{}^{\mu\nu}$, $T'_{PA}{}^{\nu}$, and T'_{PP} of the amplitudes $T_{AA}{}^{\mu\nu}$, $T_{PA}{}^{\nu}$, and T_{PP}, we obtain relations similar to (3.7)–(3.9), which have on their right-hand sides only the terms corresponding to the first term on the right-hand sides of (3.7)–(3.9). Together with (3.7)–(3.9) these may be used for obtaining relations involving integrals of the absorptive parts, similar to those derived by Fubini.[7]

We now derive relations for πN scattering, restricting the SU_3 indices in Eqs. (3.7)–(3.9) to the isospin subgroup. Separating Eq. (3.9) into equations for the amplitudes symmetric and antisymmetric in the isospin indices, we obtain

$$q_{1\mu}q_{2\nu}T_{AA}{}^{\mu\nu(-)}=C_\pi{}^2(\mu_\pi{}^2-q_1{}^2)^{-1}(\mu_\pi{}^2-q_2{}^2)^{-1}T_{PP}{}^{(-)}+q_{2\nu}F^\nu(\gamma\sigma)+q_{1\mu}q_{2\nu}R^{\mu\nu(-)}, \quad (3.15)$$

$$q_{1\mu}q_{2\nu}T_{AA}{}^{\mu\nu(+)}=C_\pi{}^2(\mu_\pi{}^2-q_1{}^2)^{-1}(\mu_\pi{}^2-q_2{}^2)^{-1}T_{PP}{}^{(+)}+(\mu_\pi{}^2-q_1{}^2)^{-1}C_\pi\mathcal{H}(\gamma\sigma)+q_{1\mu}q_{2\nu}R^{\mu\nu(+)}, \quad (3.16)$$

where we have put

$$\begin{aligned} if_{\alpha\beta\beta'}F_{\beta'}{}^\nu(\beta'\gamma\sigma) &\to \tfrac{1}{2}[\tau_\alpha,\tau_\beta]F^\nu(\gamma\sigma), \\ d_{\beta\alpha\alpha'}\mathcal{H}(\alpha'\gamma\sigma) &\to \delta_{\alpha\beta}\mathcal{H}(\gamma\sigma), \end{aligned} \quad (3.17)$$

on restricting the indices to the isospin subgroup.

Here we have written the relations for the covariant amplitudes $T_{AA}{}^{\mu\nu(\pm)}$ and have defined the amplitudes $T_{AA}{}^{\mu\nu(\pm)}$ and $T_{PP}{}^{(\pm)}$ by the decomposition

$$T_{\beta\alpha}=\delta_{\beta\alpha}T^{(+)}+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]T^{(-)}, \quad (3.18)$$

where T denotes $T_{AA}{}^{\mu\nu}$ or T_{PP}. A similar decomposition defines $R^{\mu\nu(\pm)}$.

We also write

$$\begin{aligned} T^{(\pm)}&=\bar{u}(p_f)M^{(\pm)}u(p_i), \\ M^{(\pm)}&=A^{(\pm)}+(\gamma\cdot Q)B^{(\pm)}, \end{aligned} \quad (3.19)$$

and introduce the variables

$$\begin{aligned} &\nu=q_1\cdot(p_i+p_f)/2m_N; \quad \nu_B=-q_1\cdot q_2/2m_N; \\ &s=(p_i+q_1)^2=W^2; \quad t=(p_i-p_f)^2; \quad u=(p_i-q_2)^2. \end{aligned} \quad (3.20)$$

The amplitudes T^I, M^I with definite isospin I are given in a well-known way by

$$T^{(1/2)}=T^{(+)}+2T^{(-)}; \quad T^{(1/2)}=T^{(+)}-T^{(-)}, \text{ etc.} \quad (3.21)$$

To obtain relations for the S-wave scattering lengths, we consider forward πN scattering, and put $q_1=q_2=q$. Multiplying Eqs. (3.15) and (3.16) by $(\mu_\pi{}^2-q^2)^2$ and taking the limit $\mathbf{q}\to 0$, $q_0\to 0$ in the c.m. frame, we obtain the following relations:

$$\lim \nu g_A{}^2\left(1-\frac{m^2}{p_0{}^2}\right)=\lim\{-\nu+\mu^{-4}C^2T^{(-)}\}; \quad (3.22a)$$

$$0=\mu^{-4}C^2\lim T^{(+)}+\mu^{-2}Ch(\gamma\sigma). \quad (3.22b)$$

The limits involved are $\mathbf{q}\to 0$, $q_0\to 0$. We note that the last terms on the right-hand side of (3.15) and (3.16), as well as their derivatives with respect to ν, vanish in this limit, as $R^{\mu\nu(\pm)}$ are nonsingular, so that the Schwinger term in the CR (3.5) will not contribute to relations for $T^{(\pm)}$ and $\partial T^{(\pm)}/\partial\nu$ following from (3.15) and (3.16) in the limit $\mathbf{q}\to 0$, $q_0\to 0$. The same is true when the Schwinger term is of the more general form mentioned earlier. The function $h(\gamma\sigma)$ in (3.22b) is defined by

$$h(\gamma\sigma)=\lim_{\mathbf{q}\to 0,\, q_0\to 0}\mathcal{H}(\gamma\sigma). \quad (3.23)$$

The left-hand sides of Eqs. (3.22) arise from the Born terms; we have here treated the neutron and proton masses as degenerate.[13] The zero on the left-hand side of (3.22b) arises from the limit, as $q_1=q_2=q\to 0$, of a term proportional to $(q_2-q_1)\cdot q_1/p_i{}^0p_f{}^0$. This term and its derivative with respect to ν both vanish in the limit $q\to 0$. We note that the term $\mathcal{H}$ in Eq. (3.16) is independent of ν [since from (3.13), $\mathcal{H}$ can depend only on $p_i{}^2$, $p_f{}^2$ and $(q_1-q_2)^2=t$]. One then obtains the following relations:

$$\lim T^{(-)}=0; \qquad \lim\left[\frac{\partial}{\partial\nu}T^{(-)}\right]=\frac{G^2K^2(0)}{2g_A{}^2m^2}; \quad (3.24a)$$

$$\lim T^{(+)}=\frac{-GK(0)}{mg_A}h; \quad \lim\left[\frac{\partial}{\partial\nu}T^{(+)}\right]=0, \quad (3.24b)$$

where again the limit involved is $\mathbf{q}\to 0$, $q_0\to 0$, which may alternatively be expressed as $\mathbf{q}\to 0$, $\bar{\mu}\equiv\sqrt{(q^2)}\to 0$, i.e., it is the limit of zero-energy forward scattering of pions of zero "external" mass $\bar{\mu}$. In obtaining these we have used the relation

$$C_\pi=\frac{g_Am\mu^2}{GK(0)}, \quad \text{with} \quad m=m_N, \quad \mu=\mu_\pi, \quad G=G_{NN\pi}, \quad (3.25)$$

where $K(t)$ is the pionic form factor of the nucleon for momentum transfer t.

Relations similar to (3.24) may be written for the proper parts $T_p{}^{(\pm)}$ of the forward scattering amplitudes at $\mathbf{q}=0$, $q_0=0$. For this we separate out the pole terms on each side of Eqs. (3.15) and (3.16), note that they cancel, and then take the limit $q\to 0$, $q_0\to 0$. This gives the following equations:

$$\lim T_p{}^{(-)}=0; \quad \lim\frac{\partial}{\partial\nu}T_p{}^{(-)}=\frac{G^2K^2(0)}{2g_A{}^2m^2}(1-g_A{}^2); \quad (3.26a)$$

$$\lim T_p{}^{(+)}=\frac{G^2K^2(0)}{m}-\frac{GK(0)}{mg_A}h; \quad \lim\frac{\partial}{\partial\nu}T_p{}^{(+)}=0. \quad (3.26b)$$

[13] When the intermediate baryon has a mass different from the initial baryon, the Born terms that give rise to the left-hand side of Eqs. (3.22) vanish in the limit $\mathbf{q}\to 0$, $q_0\to 0$; their derivatives also vanish. Equations (3.24) are again obtained.

To compare Eqs. (3.24) and (3.26), we note that the limit of the Born terms

$$T_{\text{imp}}^{(\pm)}(\nu,\nu_B)=\nu\frac{G^2}{2m}K^2(0)\left[\frac{1}{\nu_B-\nu}\mp\frac{1}{\nu_B+\nu}\right], \quad (3.27)$$

are given by

$$\lim T_{\text{imp}}^{(-)}=0;$$

$$\lim\frac{\partial}{\partial\nu}T_{\text{imp}}^{(-)}=\frac{G^2K^2(0)}{2m^2}=\mu^4C^{-2}g_A{}^2; \quad (3.28a)$$

$$\lim T_{\text{imp}}^{(+)}=-\frac{G^2K^2(0)}{m};\quad \lim\frac{\partial}{\partial\nu}T_{\text{imp}}^{(+)}=0, \quad (3.28b)$$

where the limit is again $\mathbf{q}\to 0$, $q_0\to 0$.

These are obtained from (3.27), on writing

$$\frac{\partial}{\partial\nu}T_{\text{imp}}^{(\pm)}(\nu)|_{\nu=0}=\lim_{\nu\to 0}\frac{[T_{\text{imp}}^{(\pm)}(\nu)-T_{\text{imp}}^{(\pm)}(0)]}{\nu}, \quad (3.29)$$

putting $\nu_B=-q^2/2m$, and taking $\mathbf{q}\to 0$, $q_0\to 0$. Note that the vanishing of $T^{(-)}$, $\partial T^{(+)}/\partial\nu$, etc., in (3.24), (3.26), and (3.28) at $\nu=0$ follows from crossing symmetry.

To obtain the scattering lengths we need the values of $T^{(\pm)}$ at the physical threshold $\nu=\mu$, $t=0$, $q^2=\mu^2$. The extrapolation of the amplitudes $T^{(\pm)}$ from $\nu=0$, $t=0$, $q^2=0$ to the physical threshold may be performed in two ways.

One may start with (3.26) and assume the following extrapolation for the amplitudes $T_p^{(\pm)}(\nu,t,q_1{}^2,q_2{}^2)$:

$$T_p^{(\pm)}(\mu,0,0,0)\approx T_p^{(\pm)}(0,0,0,0)+\mu\frac{\partial}{\partial\nu}T_p^{(\pm)}(\nu,0,0,0)|_{\nu=0}, \quad (3.30a)$$

$$T_p^{(\pm)}(\mu,0,\mu^2,\mu^2)\approx T_p^{(\pm)}(\mu,0,0,0)/K^2(0), \quad (3.30b)$$

Using Eqs. (3.26) and (3.30) and adding the exact Born terms for $q^2=\mu^2$ [which are obtained by dividing (3.27) by $K^2(0)$], we obtain $T^{(\pm)}$ at the physical threshold. Using the definitions of the S-wave scattering lengths,

$$a^{(\pm)}=\frac{m}{4\pi(m+\mu)}M^{(\pm)}\left(\mu,\frac{-\mu^2}{2m}\right); \quad (3.31)$$

$$a^{(+)}=\tfrac{1}{3}(a_1+2a_3);\quad a^{(-)}=\tfrac{1}{3}(a_1-a_3); \quad (3.32)$$

we obtain

$$a^{(-)}=\frac{G^2\mu}{8\pi m^2}\left(1+\frac{\mu}{m}\right)^{-1}\left[g_A{}^{-2}+\frac{\mu^2/4m^2}{1-\mu^2/4m^2}\right]; \quad (3.33a)$$

$$a^{(+)}=\frac{-G^2}{4\pi m}\left(1+\frac{\mu}{m}\right)^{-1}\times\left[\frac{h}{g_AGK(0)}+\frac{\mu^2/4m^2}{1-\mu^2/4m^2}\right]. \quad (3.33b)$$

Alternatively, we may start with Eqs. (3.24) and assume an extrapolation analogous to (3.30) for the full amplitudes $T^{(\pm)}$. This gives the following results:

$$a^{(-)}=\frac{G^2\mu}{8\pi m^2g_A{}^2}\left(1+\frac{\mu}{m}\right)^{-1}; \quad (3.34a)$$

$$a^{(+)}=\frac{-Gh}{4\pi m^2g_AK(0)}\left(1+\frac{\mu}{m}\right)^{-1}. \quad (3.34b)$$

As the Born terms vary fairly rapidly with ν for small ν,[14] and as the amplitudes $T_p^{(\pm)}$ vary more slowly than $T^{(\pm)}$,[15,16] it seems to be a better approximation to extrapolate $T_p^{(\pm)}$ using (3.30) and add the exact Born terms than to extrapolate $T^{(\pm)}$ directly. However, the extrapolation from $\nu=0$, $t=0$, $q^2=0$ to the physical threshold involves an extrapolation in $\bar{\mu}\equiv\sqrt{(q^2)}$ as well as in ν, and it is useful to examine quantitatively the difference between the two extrapolations.

For the experimental values of the S-wave scattering lengths, we take the estimates of Samaranayake and Woolcock (referred to as SW)[17]:

$$a_1=0.183\pm0.017;\qquad a_3=-0.109\pm0.010; \quad (3.35a)$$

$$a^{(+)}=-0.012\pm0.004;\quad a^{(-)}=0.097\pm0.007. \quad (3.35b)$$

The estimates given by other authors[18] for $a^{(-)}$ agree with (3.35b) to about 10–12%, while those for $a^{(+)}$ differ considerably. We also assume $G_{NN\pi}{}^2/4\pi\approx 14.6$, $g_A\approx -1.18$.

Equations (3.33a) and (3.34a) differ by the last term on the right-hand side of (3.33a), which is only about 0.8% of the first term. In (3.33b), the first term on the right is unknown; the left side is also not known to any degree of accuracy. However, the second term on the right-hand side of (3.33b) is seen to be comparable in

[14] Note that although the slope of the Born term in $T^{(+)}$ vanishes at $\nu=0$ (because of crossing symmetry), this slope itself varies rapidly, and gives rise to a rapid variation of the Born term in $T^{(+)}$.

[15] That $T_p^{(\pm)}$ vary more slowly than $T^{(\pm)}$ at low energies for the physical amplitudes may be seen from the experimental phase shifts. At the unphysical values of ν and of q^2 of interest here, a slow variation may be shown by assuming a dispersion relation in ν. For $q^2=\mu^2$, the absorptive parts may be evaluated in terms of total cross sections; one expects that going to $q^2=0$ should not alter the properties of the amplitudes appreciably.

[16] A quantitative study of these features and of higher order terms in the amplitudes will be reported elswehere.

[17] Y. Samaranayake and W. I. Woolcock, Phys. Rev. Letters 15, 936 (1965).

[18] See, for example, J. Hamilton and W. I. Woolcock, Rev. Mod. Phys. 35, 737 (1963). These authors give $a_1=0.171\pm0.005$; $a_3=-0.088\pm0.004$; from which we obtain $a^{(-)}=0.086\pm0.009$; $a^{(+)}=-0.0025\pm0.013$.

magnitude to the left-hand side, so that (3.33b) and (3.34b) differ significantly.

Equations (3.33) give the following expressions for a_1 and a_3:

$$a_1=\frac{G^2\mu}{4\pi m^2}\left(1+\frac{\mu}{m}\right)^{-1}\left\{g_A^{-2}-\frac{(\mu/4m)(1-\mu/m)}{1-\mu^2/4m^2}-\frac{mh}{\mu g_A GK(0)}\right\}, \quad (3.36a)$$

$$a_3=\frac{-G^2\mu}{8\pi m^2}\left(1+\frac{\mu}{m}\right)^{-1}\times\left\{g_A^{-2}+\frac{\mu/2m}{1-\mu/2m}+\frac{2mh}{\mu g_A GK(0)}\right\}. \quad (3.36b)$$

Using the experimental estimate of $a^{(+)}$ given by SW [see Eq. (3.35b)] in (3.33b) gives $h\approx 0.02K(0)$; the term with h contributes about 1–2% to a_1 and a_3 [assuming $K(0)$ to be not too different from unity] and may be neglected. Equation (3.33a) gives $a^{(-)}\approx 0.102$, which agrees reasonably well with (3.35b). In terms of a_1 and a_3, the results are $a_1\approx 0.192$, $a_3\approx -0.114$.

We finally note that using (3.34) and neglecting h gives $a^{(+)}=0$, and[19,20]

$$a_1=-2a_3=G^2\mu[4\pi g_A{}^2m^2(1+\mu/m)]^{-1}\approx 0.202. \quad (3.37)$$

Relations similar to (3.36) may be derived for the other meson-baryon scattering lengths. Further questions arise in connection with K-meson scattering lengths; these will be discussed elsewhere.

The derivation here shows the occurrence of the term $\mathcal{H}(\gamma\sigma)$ in Eq. (3.16) when q_1, q_2 are kept off the mass shell. The reason why this term occurs in our equations and not in those of some other authors[19] is analyzed in the Appendix; it is related to the ambiguity in defining off-mass-shell extensions of S-matrix elements.

We note in passing that the second equation in (3.26a), together with an unsubtracted dispersion relation for $T_p{}^{(-)}/\nu$, gives the Adler-Weisberger relation.[21]

We shall finally obtain a rough approximation for low-energy forward πN scattering, starting with the values of the πN forward scattering amplitude and its derivative (with respect to ν) at the unphysical limit $\nu=0$, $q_1{}^2=q_2{}^2=0$, which are obtained from the current algebra, and extending the extrapolation (3.30) (and the analogous one for $T^{(\pm)}$) above threshold. We shall write down the expressions resulting from (3.26) and (3.30), as well as those obtained by starting with (3.24) and extrapolating $T^{(\pm)}$, and examine whether there is any empirically significant difference between the two extrapolations.

Equations (3.26) gives the values of $\mathrm{Re}T_p{}^{(\pm)}$ and $\partial\,\mathrm{Re}T_p{}^{(\pm)}/\partial\nu$ at $\nu=0$ (for $q^2=0$) and state that $\mathrm{Im}T_p{}^{(\pm)}$ and $\partial\,\mathrm{Im}T_p{}^{(\pm)}/\partial\nu$ vanish at this point. Assuming that $\mathrm{Re}T_p{}^{(\pm)}$ are continuous and slowly varying functions of ν for small ν, we write

$$\mathrm{Re}T_p{}^{(\pm)}(\nu)\approx\mathrm{Re}T_p{}^{(\pm)}(0)+\nu\left[\frac{\partial}{\partial\nu}\mathrm{Re}T_p{}^{(\pm)}\right]_{\nu=0} \quad (3.38)$$

for small ν. As the lowest "internal" threshold, i.e., the threshold for the $(\pi+N)$ intermediate state with a massive pion, is at $\nu=\mu+\mu^2/2m$ (for $q_1{}^2=q_2{}^2=0$), the imaginary parts of the amplitudes are zero below the branch point at $\nu=\mu+\mu^2/2m$ (and become nonzero above this point), and the vanishing of $\mathrm{Im}T_p{}^{(\pm)}$ and their derivatives with respect to ν at $\nu=0$ does not give any indication of the behavior of $\mathrm{Im}T_p{}^{(\pm)}$ above the threshold. (Note that the real parts of the amplitudes are continuous at the threshold branch-point, although the imaginary parts rise from zero to a nonzero value.)

Using (3.38), assuming a relation analogous to (3.30b) for $\mathrm{Re}T_p{}^{(\pm)}$ at small ν, and adding the exact Born terms, we obtain the following low-energy approximation for the real part of the forward πN scattering amplitude $T(\nu,t)$ (with massive pions):

$$\mathrm{Re}T^{(+)}(\nu,0)=\frac{-Gh}{mg_AK(0)}-\frac{G^2}{m}\frac{\mu^4/4m^2}{\nu^2-\mu^4/4m^2}; \quad (3.39a)$$

$$\mathrm{Re}T^{(-)}(\nu,0)=\frac{G^2\nu}{2m^2g_A{}^2}+\frac{G^2\nu}{2m^2}\frac{(-\nu^2+\mu^2+\mu^4/4m^2)}{\nu^2-\mu^4/4m^2}. \quad (3.39b)$$

The alternative extrapolation starting with Eqs. (3.24) for $\mathrm{Re}T^{(\pm)}$ gives the following:

$$\mathrm{Re}T^{(+)}(\nu,0)=\frac{-Gh}{mg_AK(0)}; \quad (3.40a)$$

$$\mathrm{Re}T^{(-)}(\nu,0)=\frac{G^2\nu}{2m^2g_A{}^2}. \quad (3.40b)$$

The expressions (3.39) and (3.40) differ by the last terms in the right-hand sides of (3.39a) and (3.39b). The last term in the right-hand side of (3.39a) is comparable to the left-hand side, so that there is a significant difference between (3.39a) and (3.40a). At threshold, the

[19] Other authors who have obtained these expressions for the S-wave scattering lengths are Y. Tomozawa, Institute for Advanced Study, Princeton, New Jersey (to be published); A. P. Balachandran, M. Gundzik, and F. Nicodemi, Nuovo Cimento 44, 1257 (1966); B. Hamprecht, Cambridge University, Cambridge, England (to be published); and Ref. 20 below.

[20] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[21] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); Phys. Rev. **140**, B736 (1965); W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965).

TABLE I. Comparison of the expressions for $\mathrm{Re}T^{(\pm)}$ for low-energy πN forward scattering with experiment.[a]

Pion lab kinetic energy (in MeV)	$[\mathrm{Re}T^{(-)}]_1$	$[\mathrm{Re}T^{(-)}]_2$	$[\mathrm{Re}T^{(-)}]_{\mathrm{expt}}$	$[\mathrm{Re}T^{(+)}]_1$	$[\mathrm{Re}T^{(+)}]_2$	$[\mathrm{Re}T^{(+)}]_{\mathrm{expt}}$
0	1.45	1.46	1.24	−0.173	−0.173	0.334
6	1.34	1.52	1.13	−0.161	−0.173	0.198
10	1.27	1.56	1.06	−0.154	−0.173	0.21
15	1.19	1.615	1.0	−0.146	−0.173	0.224
20	1.12	1.67	0.94	−0.138	−0.173	0.237
25	1.04	1.725	0.845	−0.131	−0.173	0.248
31	0.96	1.785	0.75	−0.124	−0.173	0.26

[a] $[\mathrm{Re}T^{(\pm)}]_1$ are the values obtained from (3.39), and $[\mathrm{Re}T^{(\pm)}]_2$ those obtained from (3.40). $[\mathrm{Re}T^{(+)}]_1$ and $[\mathrm{Re}T^{(+)}]_2$ agree at zero energy because they were both obtained from the scattering lengths $a^{(+)}$ as given by SW (Ref. 17). $[\mathrm{Re}T^{(\pm)}]_{\mathrm{expt}}$ are obtained from the phase shifts of RWF (Ref. 22).

last term in the right-hand side of (3.39b) is negligible compared to the first term; however, at energies above about 6 MeV (pion lab kinetic energy), the difference between (3.39b) and (3.40b) becomes appreciable.

Writing the partial-wave expansions of $\mathrm{Re}T^{(\pm)}$ and keeping only the S, P, and D waves, we may express Eqs. (3.39) in the form of the following sum rules for the low-energy phase shifts:

$$\sin2\delta_{0+}+\sin2\delta_{1-}+2\sin2\delta_{1+}+2\sin2\delta_{2-}+3\sin2\delta_{2+}=\frac{qG^2}{2\pi W}\left[-\frac{\nu}{m}(1-g_A{}^{-2})+\frac{\mu^2}{m}\frac{(\nu-\mu^2/4m)}{\nu^2-\mu^4/4m^2}-\frac{h}{g_AGK(0)}\right]; \quad (3.41\mathrm{a})$$

$$\sin2\eta_{0+}+\sin2\eta_{1-}+2\sin2\eta_{1+}+2\sin2\eta_{2-}+3\sin2\eta_{2+}=\frac{qG^2}{2\pi W}\left[\frac{\nu}{2m}(1-g_A{}^{-2})-\frac{\mu^2/2m}{\nu-\mu^2/2m}-\frac{h}{g_AGK(0)}\right]. \quad (3.41\mathrm{b})$$

In these, $\delta_{l\pm}$ denote the $I=\frac{1}{2}$ phase shifts and $\eta_{l\pm}$ the $I=\frac{3}{2}$ phase shifts.

One may similarly write down the sum rules following from (3.40).

To compare these relations with experiment, we use the estimate of h obtained from the experimental value of (a_1+2a_3),[17] and the low-energy phase shifts given by the 0–350-MeV solution of Roper, Wright, and Feld (RWF).[22]

The values of $\mathrm{Re}T^{(-)}$ and $\mathrm{Re}T^{(+)}$ at low energies predicted by (3.39) and (3.40) are shown in Table I, together with the values obtained from the phase shifts of RWF. The phase shifts at 10, 15, and 25 MeV were obtained by interpolation from the values given by RWF.

As the value of h [in (3.39a) or (3.40a)] has been obtained by using the experimental value of $a^{(+)}$ (as estimated by SW), and as the latter is not known accurately, the predictions for $\mathrm{Re}T^{(+)}$ in Table I are useful only in showing the nature of the variation with energy of $\mathrm{Re}T^{(\pm)}$, rather than in giving its absolute magnitude.

The 0–350-MeV solution of RWF was obtained by an over-all fit to the data between 0 and 350 MeV, and is not expected to give accurate values of the low-energy phase shifts. Thus, for the scattering amplitudes at threshold, these phase shifts give $T^{(-)}\approx1.243$ and $T^{(+)}\approx0.335$, whereas the estimates of SW for the scattering lengths[17] (which were obtained from integrals over total cross sections and are expected to be more reliable) give $T^{(-)}\approx1.46$, $T^{(+)}\approx-0.173$ at threshold. The threshold value of $T^{(+)}$ obtained from the estimates of RWF differs even in sign from that obtained by SW; one would therefore not expect the predictions for $\mathrm{Re}T^{(+)}$ obtained from the phase shifts of RWF to be reliable at energies near the threshold. However, at energies not too close to threshold, the qualitative variation of $\mathrm{Re}T^{(+)}$ with energy is probably given correctly by the estimates of RWF. For $\mathrm{Re}T^{(-)}$, the estimates would be more reliable, although not accurate numerically. We shall compare below the nature of the variation with energy of $\mathrm{Re}T^{(+)}$ and $\mathrm{Re}T^{(-)}$ as predicted by (3.39) and (3.40) with that of the experimental values obtained from the estimates of RWF.

Table I shows that $\mathrm{Re}T^{(-)}$ as predicted by (3.39b) decreases with increasing energy (at low energies), as do the experimental values, whereas (3.40b) predicts that $\mathrm{Re}T^{(-)}$ should increase linearly with energy. Also, the numerical values predicted by (3.39b) are closer to the experimental values than the predictions of (3.40b).

Equation (3.39a) predicts that $\mathrm{Re}T^{(+)}$ should increase slowly with energy, whereas (3.40a) predicts that it should be constant at low energies. Above 6 MeV, the experimental values increase slowly with increasing energy, in agreement with (3.39a); at lower energies the predictions obtained from the phase shifts of RWF are probably not reliable, as observed earlier.

More accurate values of the phase shifts in the low-energy region are needed before definite conclusions can be reached about the quantitative agreement of the

[22] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965).

predictions with experiment. However, the available estimates already suggest, for the amplitude $T^{(-)}$, that with the simple method of extrapolation used here, the procedure of extrapolating the proper part of the amplitude linearly and adding the exact Born term gives a better approximation than extrapolating the total amplitude linearly. As stated earlier, this may be expected, because the Born terms vary relatively rapidly at low energies, and the proper parts of the amplitudes vary more slowly with energy than the full amplitudes.

Further results on meson-baryon scattering will be discussed in subsequent papers.[23]

IV. CONCLUSIONS

In this paper we have discussed how the generalized Ward-Takahashi identities for a current algebra generated by conserved and partially conserved currents may be used for deriving relations for physical amplitudes and their absorptive parts. We have illustrated in the simple example of meson-baryon scattering how the exact consequences of the current commutation relations and the PCAC hypothesis, which give the scattering amplitude and its derivative in an unphysical limit, may be used as boundary conditions starting from which one may obtain the scattering lengths and simple low-energy relations for a physical amplitude.

We have examined some questions that arise in making an off-mass-shell extension of an S-matrix element, and have shown how using the PCAC hypothesis at different stages of a reduction procedure can give different results when some of the momenta are off the mass shell.[24] We suggest that in order to make explicit the questions arising in off-mass-shell limits, one should reduce the S-matrix element completely to the Fourier transform of a vacuum expectation value before using relations between local operators such as the PCAC hypothesis and current commutation relations. This ensures that all the terms that should occur on using such relations are taken into account. This will be further discussed in a separate work.

In subsequent papers we shall discuss further results for two-body reactions and for the production of low-energy mesons[25] in a general process involving a multiparticle final state.

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APPENDIX

In this Appendix we examine why the extra term $\mathcal{H}$ obtained by us on the right-hand side of (3.8), (3.9), and (3.16) does not appear in the derivation of some other authors.[19] We analyze this explicitly as it illustrates the questions arising in off-mass-shell limits.

We keep the baryons on the mass shell throughout, and therefore leave them in the state vectors for the initial and final states.

Using the PCAC hypothesis, the matrix element for πN scattering may be written as

$$\langle p_f q_2 \text{ out} | p_i q_1 \text{ in}\rangle - \delta_{p_f p_i}\delta_{q_1 q_2}$$

$$= C^{-1} i[2q_2{}^0]^{-1/2}\int d^4x\, e^{iq_2\cdot x}\mathcal{K}_x \times\langle p_f|\partial_\mu \mathcal{A}^\mu(x)|q_1,p_i \text{ in}\rangle \quad \text{(A1)}$$

$$= C^{-1} i^2[4q_1{}^0 q_2{}^0]^{-1/2}\int d^4x d^4y\, e^{iq_2\cdot x}\, e^{-iq_1\cdot y}\mathcal{K}_x\mathcal{K}_y \times\langle p_f|[\partial_\mu \mathcal{A}^\mu(x),\varphi_1(y)]\theta(x_0-y_0)|p_i\rangle, \quad \text{(A2)}$$

where $\mathcal{K}_x$ is the Klein-Gordon operator $(\Box_x{}^2+\mu^2)$.

Removing the derivative from the matrix element in (A1) gives

$$-iq_{2\mu} C^{-1} i[2q_2{}^0]^{-1/2}\int d^4x\, e^{iq_2\cdot x}\mathcal{K}_x \times\langle p_f|\mathcal{A}^\mu(x)|q_1,p_i \text{ in}\rangle \quad \text{(A3)}$$

$$= -iq_{2\mu} C^{-1} i^2[4q_1{}^0 q_2{}^0]^{-1/2} \times\int d^4x d^4y\, e^{iq_2\cdot x}\, e^{-iq_1\cdot y}\mathcal{K}_x\mathcal{K}_y \times\langle p_f|[\mathcal{A}^\mu(x),\varphi_1(y)]\theta(x_0-y_0)|p_i\rangle. \quad \text{(A4)}$$

On the other hand, taking the derivative out in (A2) and integrating by parts gives, in addition to (A4), a term

$$-C^{-1} i^2[4q_1{}^0 q_2{}^0]^{-1/2}\int d^4x d^4y \times e^{iq_2\cdot x}\, e^{-iq_1\cdot y}(\mu_2{}^2-q_2{}^2)(\mu_1{}^2-q_1{}^2) \times\langle p_f|[\mathcal{A}_2{}^0(x),\varphi_1(y)]\delta(x_0-y_0)|p_i\rangle. \quad \text{(A5)}$$

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[23] Results on meson-baryon scattering, using different methods, have been obtained by A. P. Balachandran, M. Gundzik, and F. Nicodemi (to be published).

[24] We are grateful to Dr. D. F. Dell'Antonio for a discussion of these questions.

[25] V. S. Mathur and L. K. Pandit have mentioned the possibility of obtaining relations for soft-pion emission using the Ward-Takahashi identity; see V. S. Mathur and L. K. Pandit, Phys. Rev. **147**, 965 (1966).

This gives exactly the additional term $\mathcal{H}$ in our equations (3.8), (3.9), and (3.16).

This additional term did not appear in the first derivation [in which ∂_μ was removed from the matrix element in (A1)] because q_1 was there kept on the mass shell, by virtue of being kept in the state vector. The additional term (A5) vanishes when q_1 (or q_2) is on the mass shell; this happens because (A5) with the factor $(\mu_1^2-q_1^2)(\mu_2^2-q_2^2)$ removed does not have poles at $q_1^2=\mu_1^2$ and $q_2^2=\mu_2^2$, in contrast to the corresponding function with $\delta(x_0-y_0)$ replaced by $\theta(x_0-y_0)$.

Thus we see that if the momenta q_1 and q_2 are to be later taken off the mass shell, using the PCAC hypothesis at the two different stages of the reduction procedure considered above gives results differing by the nonvanishing term (A5), i.e., the two procedures give two different off-mass-shell continuations of the S-matrix element under consideration.

In order to specify which continuation is to be taken in such an off-mass-shell limit, an additional prescription is needed. We suggest the prescription that all those terms should be kept which arise if we reduce the S-matrix element completely (to the Fourier transform of a vacuum expectation value) before using divergence conditions (like current conservation or the PCAC hypothesis) and current commutation relations. In a process involving photons, for instance, this insures that no contribution is omitted in which a soft photon is emitted by some external off-mass-shell charged particle in the process.