The fundamental theorem on the connection between spin and statistics

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It appears that all known particles obey either Bose or Fermi statistics; and that integral spin particles like the photon and the pion obey Bose statistics while half-integral spin particles like the electron and the nucleon obey Fermi statistics. It would be very satisfying to be able to deduce the observed connection between spin and statistics from the basic postulates of quantum theory of fields. In the present paper we state and prove a theorem within a general quantum theory formulation asserting this relation between spin and statistics.

There has been previous work on this question, notably by W. Pauli (1). The theorem of Pauli asserts the observed connection of spin and statistics within the framework of relativistic quantum theory. As such it is not applicable to a nonrelativistic situation, say electrons in an atom or in a metal. Even within the framework of relativistic theories Pauli had to make the technical assumption that the fields belong to finite-dimensional representations of the Lorentz group. The theorem of Pauli has been refined and transcribed into the axiomatic framework in recent years, but these two limitations have remained.

In the present work we show that a specific formulation of the principle of symmetry between emission and absorption processes characteristic of quantum theories already leads to the observed spin-statistics relation in all cases.

1. *The quantum-theory framework*

Interaction processes involving the photon or the electron have always exhibited a symmetry between emission and absorption. The interaction responsible for the emission of a photon implies an interaction of the same strength responsible for the absorption of the photon. The positron emission beta interaction automatically includes an equally strong interaction for orbital electron capture. We should include this as a basic property of quantum field theory, whether it be a relativistic field theory or a nonrelativistic field theory. We shall refer to a precise mathematical statement of this basic symmetry as the *S*-principle.

The conventional quantization of relativistic finite-component fields include both creation and destruction operators in the same field. As a consequence the use of local finite-component relativistic fields would go a long way to satisfy such a principle. But clearly the symmetry between emission and absorption should be true even for the nonrelativistic limit of such a theory though in this limit the field decomposes locally into creation and annihilation parts.

We shall be concerned with Lagrangian field theories and make explicit use of the Action principle. The Action function is the space-time integral of a Lagrangian density. Without loss of generality we may use first order derivatives only in the Lagrangian density. The general expression would then be of the form:

$$L(x)=\frac{i}{2}\Gamma_{rs}\left\{\psi_r^\dagger(x)\frac{\partial\psi_s(x)}{\partial t}-\frac{\partial\psi_r^\dagger(x)}{\partial t}\psi_s(x)\right\}+H(x)$$

where $H(x)$ is independent of time derivatives. We shall require the following: The Action function shall remain invariant under the change

$$\psi(x)\rightleftharpoons\varphi(-x)$$

where $\varphi(x)$ is a field whose components are linear combinations of the components of $\psi^\dagger(x)$ such that the field $\varphi(x)$ transform in the same manner as $\psi(x)$:

$$\varphi_r(x)=\psi_s^\dagger(x)\,E_{sr};\quad E^2=1$$

Since the Action is unchanged under the transformation

$$\psi_r(x)\to\psi_s^\dagger(-x)\,E_{sr},$$

the equations of motion would remain unchanged.

For a relativistic theory Γ will be one of a set of four vector matrices Γ^μ, but no such restrictions are imposed in the general case. Only rotational invariance is demanded of the Lagrangian density.

Both commutation relations and equations of motion for the fields can be obtained from the Weiss-Schwinger Action principle (2):

$$i\delta\psi_r(x)=[\psi_r(x),\delta A],$$

$$A=\int d^4x\,L(x)$$

For Bose fields we take $\delta\psi$ to commute with the field operators and for Fermi fields we take $\delta\psi$ to anticommute with the field operators.

The Lagrangian density symmetrized in accordance with the S-principle would read:

$$L(x) = \frac{i}{4}(E\Gamma)_{rs}\left\{\varphi_r(x)\frac{\partial\psi_s(x)}{\partial t} - \frac{\partial\varphi_r(x)}{\partial t}\psi_s(x) - \psi_r(x)\frac{\partial\varphi_s(x)}{\partial t} + \frac{\partial\psi_r(x)}{\partial t}\varphi_s(x)\right\} + H(x)$$

where $H(x)$ is suitably symmetrized with respect of the S-principle. As far as the commutation relations are concerned the precise form of $H(x)$ need not be known; only the coefficients of the terms involving the time derivatives of the fields enter the commutation relations.

2. *The connection between spin and statistics*

For Bose fields we get from the Action principle and the S-symmetrized Lagrangian density:

$$\tfrac{1}{2}\delta(x^0 - y^0)\{(E\Gamma)_{ns} + (E\Gamma)_{sn}\}[\varphi_r(x), \psi_n(y)] = \delta_{rs}\delta(x - y)$$

Hence the matrix $(E\Gamma)$ should be symmetric for Bose fields. Similarly, for Fermi fields we get the anticommutation rules

$$\tfrac{1}{2}\delta(x^0 - y^0)\{(E\Gamma)_{ns} - (E\Gamma)_{sn}\}\{\varphi_r(x), \psi_n(y)\} = \delta_{rs}\delta(x - y)$$

Consequently $(E\Gamma)$ should be antisymmetric for Fermi fields. The connection between spin and statistics is now reduced to the purely geometric problems of determining the symmetry properties of the matrix $(E\Gamma)$.

The transformation reversing the sign of the space-time coordinates and preserving the equations of motion is called strong reflection within the framework of finite-component relativistic field theory (3). We shall have to consider strong reflection not only for those cases but also for the case of infinite-component fields as well as for nonrelativistic fields. We shall discuss these cases in turn.

Relativistic finite-component fields. In this case the strong reflection transformation can be obtained as a real element of the complex Lorentz group and we write for the Dirac field (in the Majorana representation);

$$\varphi_r(x) = \psi_s^{\dagger}(x)(\gamma_5)_{sr}$$

so that

$$E = \gamma_5$$

and

$$E\Gamma^{\mu} = \gamma_5\gamma^0\gamma^{\mu}$$

Hence the matrix $(E\Gamma)$ is antisymmetric and consequently a Dirac field must obey Fermi statistics.

Any finite collection of finite-dimensional representations of the Lorentz group can be viewed as a totally symmetric multispinor. The four-vector matrices Γ^μ for such a multispinor can be expanded in terms of a collection of basic four-vector matrices (4). The matrix E is then the product of one γ_5 for each multispinor index. It can then be easily seen that $(E\Gamma)$ would be symmetric for even rank multispinors and antisymmetric for odd rank multispinors. But even rank multispinors describe integral spin fields and odd rank multispinors describe half-integral spin fields.

We have thus deduced the observed connection between spin and statistics for these fields.

Nonrelativistic fields. In this case the strong reflection transformation is not an element of any familiar group but it can be implemented in the form

$$\varphi_r(x) = (i\sigma_2)_{rs}\psi_s^\dagger(-x) = C_{rs}\psi_s^\dagger(-x)$$

for a spin $\frac{1}{2}$ field. Any finite-component field can be obtained as a multispinor of finite rank. Since the Cartan matrix C is antisymmetric it follows that the strong reflection matrix E is symmetric for integral spins and antisymmetric for half-integral spins. Consequently we obtain the observed connection between spin and statistics for nonrelativistic finite-component fields.

Infinite-component fields. The deduction in these cases proceeds essentially along the same lines. By considering the transformation properties of the geometric strong reflection matrix E with respect to rotations we arrive again at the conclusion that $(E\Gamma)$ is symmetric for integral spin fields and antisymmetric for half-integral spin fields.

It has been pointed out that it appears possible to relax the spin-statistics connection for infinite-component relativistic fields. This is of the same nature as the apparent possibility to relax this connection for nonrelativistic fields. But in either case, if we impose the S-principle symmetrization with respect to strong reflection this freedom to relax the spin-statistics connection disappears.

3. *Remarks*

The fundamental connection between spin and statistics has been derived here from the basic principles of Lagrangian theories, and applies equally well to all quantum field theories in which "spin" is defined: namely all rotationally invariant field theories. No special mention is made of fields with only time-like or light-like states since such a qualification is unnecessary. In particular it holds for the quantum theory of tachyon (faster-than-light) fields and for more complicated wave fields which contain time-like and tachyon-like solutions. It is very satisfying to be able to eliminate the technical restrictions implicit in the statement of Pauli's theorem.

Since no requirement of positivity of the energy of the field was used the present derivations hold without modification for fields which describe both positive and negative energy particles (5).

The most interesting application of the theorem is to nonrelativistic systems: strong reflection and spin-statistics connection should apply equally well to these cases.

References

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4. K. A. Johnson & E. C. G. Sudarshan, Ann. Physics *13*, 126 (1961).
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Discussion

Fronsdal

I completely agree with you that it is extremely difficult to violate the relation between spin and statistics in an infinite-component field theory, and still expect to get something that is reasonable. I reached this conclusion in a slightly different way from yours and if I take one minute, I think that is enough to explain it.

If you start off with a finite-component field theory, given by a local Lagrangian, which means that you have an equation of motion which is a local differential equation, then you get, in the usual way, the spin and statistics theorem. If you would introduce into this differential equation a term which looks like $\varepsilon(p_0)$, the sign of p_0, something which would bring a very strong non-locality into the theory, then you could not make such a conclusion about the connection between spin and statistics. Now, if you start from an infinite-component field theory, introducing the wrong connection between spin and statistics, and deduce the equations of motion for the states with a definite value of the spin, then you find exactly such equations of motion. You find equations of motion involving $\varepsilon(p_0)$, so it is in some sense a very non-local theory, independent of questions of commutation relations and so on.

Matthews

Just a comment. You are essentially in the Lagrangian formalism and when you restrict yourself to infinite-component Lagrangians, then you run into this other problem of the space-like solutions which you discussed yesterday. Indeed, all these things are very much interconnected.

Sudarshan

That is true. This gives me an opportunity to mention something I forgot. The objection that this makes use of the Lagrangian formalism and that it does not, for example, make use of methods used by Burgoyne and other people who try to deduce this connection in general field theory is well taken; but it is quite difficult to do this because there are space-like solutions and all the usual future tube analytic properties for the Wightman functions and the spectral conditions are violated. So it is not easy to adapt immediately the same arguments for this more general situation. I have looked at the case of the space-like solutions by themselves and also in connection with simpler infinite-component wave equations, just to see if the same kind of theorem would apply in those cases, and as far as I have been able to make out, there is really no difference, whether the masses are purely imaginary or real, because they come in the dynamical part H of the Lagrangian and not in the kinematic part which really determines the commutation relations. So this method seems to be equally good for all kinds of wave equations or all kinds of field theories including the new formulation which I presented yesterday, which contains both positive and negative energy particles.

Matthews

If you abandon the Lagrangians, but stick to the requirement of causality, that is to say that your commutators or anti-commutators vanish for space-like distances, then this gives you a very powerful restriction. This restriction is very intimately connected with all these problems of TCP, antiparticles and spin and statistics.

Nambu

As was mentioned by Fubini-Sugawara recently created a new theory in which there was only energy and momentum tensor and no Lagrangian. At first I was terribly excited but lately Sugawara himself has found out that it is equivalent to a Lagrangian theory.

Domokos

I have a question to Nambu. In the Sugawara theory when you go over to the σ-field representation, is it really an equivalence transformation or something tricky like when you are trying to go between various representations in superconductivity theory? Is it a canonical transformation which brings you from the current, the V, A picture, to the σ-picture?

Nambu

I do not know. The σ-model is one realization of the current algebra. It may have a degenerate vacuum or a unique vacuum. We don't even know whether it has particle-like states.

Haag

I would comment on Fronsdal's question or suggestion which may be somewhat useful. Assume that there are no space-like momentum vectors in the theory. Well, then it seems that you need one more condition in order to obtain TCP, spin and statistics, no matter whether you use finite-component fields or not, and that condition would be that the total number of states that have bounded energy and are also localized in a finite volume are only finite in number. If you impose such a condition, which seems to be reasonable physically, then probably most of the general results of a Wightman type field theory will persist. But without such a condition, if you allow infinitely many states within a finite volume in position space and bounded energy, then the connection is too loose, then you have more freedom.

Matthews

I want to make a remark about symmetrizing a Lagrangian. The simplest well-known case is the Majorana equation, I mean the infinite Majorana-Gel'fand equation where you have only positive frequencies. Then ψ will be a field which annihilates particles; $\bar{\psi}^{\dagger}$ will create the same particles, and if you want to introduce antiparticles you have got to do it by hand with two new fields φ and $\varphi^{\dagger}$. You made some remark which simply implies that you are already there. That is not true.

Sudarshan

I do not think I have made that statement but let me make my statement precise. It is occasionally stated that in the Majorana equation there are no negative frequency solutions, which is true. However, if you consider the field theory of the Majorana equation, it would necessarily contain both positive and negative frequency operators. There is no guarantee, of course, that they describe one kind of particles or two kinds of particles, but that is very much like quantizing the Schrödinger equation. If you want to quantize the Schrödinger equation and have both particles and antiparticles, you have to introduce the antiparticles explicitly "by hand".

Matthews

No, I am sorry. The field ψ, if you expand it out in terms of creation operators, will give $a(k)$. You then take its hermitian conjugates and those automatically contain $a^\dagger(k)$. They create particles and then the negative frequency part will annihilate the same particles.

Sudarshan

So do not take the hermitian conjugate! Instead of taking the hermitian conjugate field, let us take operators which satisfy the same equation as the hermitian conjugate. For the Majorana equation we choose as Lagrangian density

$$\mathcal{L}(x) = \{\chi^\dagger(i\Gamma^\mu\partial_\mu - \varkappa)\psi\} + \{\psi^\dagger(i\Gamma^\mu\partial_\mu - \varkappa)\chi\}$$

Then, as long as $\chi \neq \psi$ we describe two different kinds of particles but the Lagrangian is hermitian.

Matthews

But an hermitian conjugate is a mathematical operation. In a theory you cannot give an order that it cannot be taken!