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Interaction between classical and quantum systems: A new approach to quantum measurement. I

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We introduce an unconventional approach to the measurement problem in quantum mechanics: we treat the apparatus as a classical system belonging to the macroworld. To describe the quantum measurement process we must couple the classical apparatus to the quantum system. In this paper we explain how this is to be done: we embed the classical apparatus into a larger quantum-mechanical structure, making use of a superselection principle. The apparatus can now be coupled to the quantum system in a straightforward manner. We discuss what constraints should be placed on the coupling so that an interpretation of the interaction as a measurement results. We require that unambiguous information of the values of a quantum observable should be transferred to the variables of the apparatus. We also require that the apparatus should retain its classical identity. This latter requirement is formulated as a principle of integrity, in both weak and strong forms.

I. INTRODUCTION

The structure and formulation of quantum-mechanical systems is by now well understood. The interaction of a number of simple quantum systems with each other has been examined repeatedly and is in remarkable quantitative agreement with experimental results. However, there is one area which even now gives rise to controversy and heated argumentation, namely the problem of measurement and the interpretation of quantum mechanics.[1] The interaction between the microcosm and the macroworld is quite simply not fully understood. In this paper we wish to discuss some topics connected with measurement.

A measurement results from the interaction between a special system (the "apparatus") and the original system in question (the "system"). In our considerations the system would be a purely quantum-mechanical system of suitably simple structure and obeying the well-known laws of quantum mechanics. In other applications, for example cosmology, the system will be a general-relativistic system. The point of view we wish to advance is that a piece of apparatus can, *and should*, be described by the laws of classical mechanics. The apparatus one uses in a laboratory, for example, belongs to the macroscopic world. The macroscopic world is apparently so well described by classical physics, be it of a statistical or a deterministic nature, that one can say it is indeed classical. Thus we shall choose to discuss and describe the apparatus by means of classical physics. The apparatus must be not only classical, but also of a simple enough structure that its relevant configurations can be easily parametrized and these parameters related to appropriate properties of the observed system: The process of making this correspondence is the calibration of the apparatus. Moreover, the configurations of the apparatus must be relatively stable on the one hand so that it can be read, while there should be selective minimal inertia to change of configuration so that the system variables can influence, and reflect themselves in the change of, the configuration of the apparatus. It is understood that we have at our disposal the ability for making and reading pointers, of setting the apparatus into preselected standard configurations, and shielding the sensitive apparatus from stray and unwanted influences.

The crux of the measuring process, then, consists in the ability to join together the apparatus and system into a single complex system and then to separate them into two. This process of coupling and decoupling is a legal process and is only approximately carried out by such loosely defined physical processes as spatial juxtaposition and separation. Primarily it amounts to the build-up and the destruction of correlations by a deliberate act.

A true piece of apparatus is, in most cases, a very complicated mechanical system. For this reason we shall not, in the present work, attempt to produce a model of classical physics for a particular piece of apparatus. Rather, we are more interested for the present in understanding the principles involved. For this reason the examples we consider may appear, at first glance, to be of a trivial nature.

For our purposes a "measurement" is achieved if unique information of the value of a quantum "system" observable can be transferred into the classical apparatus in an unambiguous fashion to be read off later by an observer.[2]

If we are to successfully use such an apparatus to measure properties of a quantum system, it is clear that we must first understand how to engineer interactions between classical systems and quantal systems. It is the purpose of this paper to pursue further an approach to this problem presented previously by Sudarshan.[3] In this approach we embed, in a highly nontrivial way, the classical system into a quantum system with twice the number of degrees of freedom. However, this quantum (-enlarged) system is richer than is allowed by measurements on the primitive classical system. The classical system *per se* is a "section" of the quantum (-enlarged) system with only a subset of dynamical variables being perceived. There is a richness that is inherent to the system which is ignored in the classical perception. This richness corresponds to the existence of dynamical variables which cannot be observed, which remain "hidden."

II. CLASSICAL SYSTEMS AS QUANTUM SYSTEMS WITH SUPERSELECTION

At first glance it may seem that the structures of quantum and classical mechanics are very different. Indeed, in quantum mechanics not all of the dynamical variables commute, whereas in classical mechanics they form a commuting set. Another difference, related to the first, is that one can superpose coherently two pure states to form another pure state in quantum mechanics, but this cannot be done in classical mechanics. Here a pure state is characterized by exact values for *all* the dynamical variables, and so any combination of such states must fail to satisfy the criterion for purity. In the usual formulation of classical mechanics it is not possible to talk about superposed *states.*

However, it is precisely these differences which show us how to embed a classical system in a quantum system. There are situations in the formulation of quantum mechanics where coherent linear combinations of pure states are not of their own right pure states. This occurs when a theory possesses superselection rules. The superselecting operators decompose the Hilbert space of states such that no observable operator can map states in one eigenspace into a different eigenspace. Furthermore, in a superposition over different superselected eigenspaces the relative phases between states belonging to different eigenspaces are not measurable.

It is possible to view superselection as a statement about unobservable operators. In the present context, namely, examining a noncommutative structure with a view to using it in the description of a classical system, we want the algebra of the observable operators to be commutative. We shall use superselection to achieve this result, albeit a nonconventional form of superselection is used.[4] In this section we shall outline this procedure.

Consider first a simple classical system. In the Hamiltonian formulation, a classical system with n degrees of freedom is characterized by n coordinates $(q_1, \ldots, q_n)$ and n conjugate momenta $(p_1, \ldots, p_n)$. The pure states of the system are given by specifying a point in phase space, i.e., we specify a value for each of the n pairs of canonical coordinates $(q_1, p_1), \ldots, (q_n, p_n)$. This defines the kinematics. To discuss the dynamics of the system one needs to write down the Hamiltonian function, $H(q,p)$, of the system. Then, Hamilton's form of the equations of motion is

$$\dot{q}_i = \frac{\partial}{\partial p_i} H(q,p)\,, \tag{2.1}$$

$$\dot{p}_i = -\frac{\partial}{\partial q_i} H(q,p)\,. \tag{2.2}$$

By using the Poisson brackets

$$\{X, Y\} \equiv \sum_i \left(\frac{\partial X}{\partial q_i} \frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial q_i} \frac{\partial X}{\partial p_i} \right), \tag{2.3}$$

these equations can be rewritten as

$$\dot{q}_i = \{q_i, H\}\,, \quad \dot{p}_i = \{p_i, H\}\,, \tag{2.4}$$

where the apparent asymmetry, due to the minus sign in (2.2), has vanished. Once the initial conditions are given in the form

$$\begin{aligned} q_i(t=0) &= q_i^0\,, \\ p_i(t=0) &= p_i^0\,, \end{aligned} \tag{2.5}$$

the dynamical evolution of the system is determined for all future times, so long as the system is not disturbed in any way.

We are now in a position to describe the quantum system in which we find this classical system embedded. We shall follow a constructive method to illustrate the choice. Let

$$\omega \equiv \{\omega^1, \ldots, \omega^{2n}\} = \{q_1, \ldots, q_n, p_1, \ldots, p_n\}\,. \tag{2.6}$$

These were the dynamical variables of the classical system. We now view them as a set of commuting operators, acting on a Hilbert space of vectors. We introduce operators

$$\pi \equiv \{\pi^1, \ldots, \pi^{2n}\}\,, \tag{2.7}$$

which are conjugate to the ω operators, with respect to commutation. That is to say,

$$[\omega^\mu, \pi^\nu] \equiv \omega^\mu \pi^\nu - \pi^\nu \omega^\mu = i\hbar\delta^{\mu\nu}\,, \tag{2.8}$$

and we will henceforth use natural units with $\hbar \equiv 1$. (This sense of conjugacy is not to be confused with the conjugacy concept in the Hamiltonian formulation of classical mechanics. There, conjugacy was with respect to the Poisson bracket, so that

$$\{q_i, p_j\} = \delta_{ij}.$$

Our conjugacy, on the other hand is defined with respect to commutation.) A representation of the π operators which is an aid to visualizing this method is

$$\pi^\mu = -i\frac{\partial}{\partial\omega^\mu}. \tag{2.9}$$

So we now have an algebra of dynamical variables which consists of $2n$ pairs of canonically conjugate operators. We shall regard *the set* $\{\omega^1, \ldots, \omega^{2n}\}$ *as a set of superselecting operators* so that the conjugate "momenta" π^μ are unobservable operators. The operators ω^μ are, thus, observable operators, and they form a commuting set. Since $\mathcal{A}$, the algebra of observables, includes all functions of the observables, it is clear that an operator belonging to $\mathcal{A}$ is observable if and only if it is independent of the unobservable operators π^μ.

The operators of $\mathcal{A}$ act on a Hilbert space of vectors, $\mathfrak{h}$. We shall concentrate on the observable operators ω^μ, as they are of primary importance to us. These operators possess a continuous spectrum. Following conventional treatments we will consider the space $\mathfrak{h}$ to be extended so that it contains also the unnormalizable eigenvectors for ω^μ. (The space of states $\mathfrak{h}$ will thus no longer be a Hilbert space.) If we denote such eigenvectors by $|\omega_0\rangle$, where

$$\omega^\mu|\omega_0\rangle = \omega_0^\mu|\omega_0\rangle, \tag{2.10}$$

then the ω representative of this state is

$$\langle\omega|\omega_0\rangle \equiv \prod_\mu \delta(\omega^\mu - \omega_0^\mu) = \delta(\omega^1-\omega_0^1)\delta(\omega^2-\omega_0^2)\cdots\delta(\omega^{2n}-\omega_0^{2n}). \tag{2.11}$$

These are the wave functions corresponding to an eigenstate of ω. Now, since the different eigenspaces of the ω^μ are superselected, we can see that to the extent direct observation is concerned, $\mathfrak{h}$ actually decomposes into a direct sum of disjoint one-dimensional state spaces corresponding to precise values for all of the observables ω^μ.

The specification of the state of the system in the Heisenberg picture can be viewed as the analog of the initial values seen in the classical formulation. This is made even more transparent if as the state of the system we choose an eigenstate of ω, corresponding to the initial values of Eqs. (2.5),

$$|\psi\rangle = |\omega_0\rangle \equiv |q^0, p^0\rangle. \tag{2.12}$$

The dynamics of the quantum system must also be specified. We note, in this regard, the following result: If we define

$$H_{\rm op} = i\sum_j\left[\frac{\partial H(q,p)}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial H(q,p)}{\partial p_j}\frac{\partial}{\partial q_j}\right], \tag{2.13}$$

then the equations of motion (2.4) appear as

$$\dot{q}_j = -i[q_j, H_{\rm op}]; \quad \dot{p}_j = -i[p_j, H_{\rm op}]. \tag{2.14}$$

These equations are very suggestive of the Heisenberg picture equations of motion. For this reason we will discuss, in this section, the time development of the quantum system in the Heisenberg picture, and treat the Schrödinger picture viewpoint later.

The time development of the operators of the quantum theory is defined in the usual manner, $A(0) \equiv A$ and

$$A(t) = e^{i\mathcal{H}t}A(0)e^{-i\mathcal{H}t}. \tag{2.15}$$

The equation of motion for $A(t)$ is

$$\dot{A}(t) = -i[A(t), \mathcal{H}].$$

Here $\mathcal{H}$ is the Hamiltonian operator of the model. Using (2.9) we may rewrite (2.13) in the form

$$\mathcal{H} = \frac{\partial H(\omega)}{\partial\omega^\nu}\epsilon^{\mu\nu}\pi^\mu, \tag{2.16}$$

with $\epsilon^{\mu\nu} \equiv \{\omega^\mu, \omega^\nu\}$ and $H(\omega)$ the Hamiltonian function of the classical theory. Then the equations of motion for $\omega^\mu(t)$ will be just Eqs. (2.14), and they can be written as

$$\dot{\omega}^\mu(t) = -i[\omega^\mu(t), \mathcal{H}]. \tag{2.17}$$

We also have a set of equations of motion for the unobservable operators $\pi^\mu(t)$,

$$\dot{\pi}^\mu(t) = -i[\pi^\mu(t), \mathcal{H}]. \tag{2.18}$$

Clearly, the dynamics of the *state of the system* is now determined for all future times, if the system is left undisturbed.

The quantum theory we have constructed above is highly nonconventional in character, so some qualifying comments are necessary. We should first note that the Hamiltonian operator (2.16), and hence also the time evolution operator, is not an observable operator. This might appear strange at first sight, as we normally associate the Hamiltonian with the energy operator. We must remember that in the theory constructed the energy function (or operator) of the physical sector is $H(\omega)$ and not $\mathcal{H}$ as given in (2.16). Thus there should be no confusion over this point.

Although the Hamiltonian operator $\mathcal{H}$ is functionally dependent on the unobservable operators π^μ, this dependence is strictly regulated. $\mathcal{H}$ is linear in π^μ. Thus, as we can see from their equations of motion, the $\omega^\mu(t)$ operators will remain independent of the operators π^μ for all times, i.e., they remain in the observable subalgebra of the algebra of dynamical variables for all times. As a corollary to this result we note that for all times t the values of all the observables of the theory can be simultaneously specified.

In the construction of the above theory we appealed to superselection to achieve a commutative algebra of observables. As we noted, the state space upon which our observables act is a direct sum of one-dimensional superselected subspaces. Thus, in *any* superposition over states the relative phases are not measurable, and so the superposition of two distinct pure states is not a pure state.

However, this usage of superselection is not the same as the conventional superselection rules.[4] It follows in the above scheme that no observable can have nonzero matrix elements between any two distinct states: It is trivial to observe that this is indeed true for all functions of ω, i.e., all "phase functions." The reproduction of the classical equations of motion led us inexorably to the Hamiltonian operator (2.16). This operator is *not* observable.

If the Hamiltonian were an observable, the time development operator would be also. It would then follow that the time development operator could not change the system from one superselected subspace to another: Consequently, the labels of these subspaces would be constants of the motion. We would have a superselection rule.

In the present case the energy function $H(\omega)$ is a phase function but the Hamiltonian operator, and hence also the time development operator, is an unobservable operator. Consequently, we can, and do, have transitions from one superselected subspace to another in the course of time. We have motions which change the labels of the superselected subspaces: Thus if a system is described at time t, by a state belonging to one superselected subspace, at a later time t_2 the system will be represented by a state belonging to a different subspace. This point will be illustrated in the examples treated in the next section.

III. SOME SIMPLE EXAMPLES

In the preceding section we introduced a general method for embedding a classical-mechanical system in a quantum-mechanical system. It is instructive at this stage to examine some simple classical-mechanical systems in the light of this embedding procedure. We shall consider very elementary problems, namely that of a classical freely moving particle, that of a particle moving in a conservative field of force, and in particular a simple harmonic oscillator.

Consider first the case of the freely moving classical particle. The conjugate dynamical variables are chosen to be $\vec{q}=(q_1,q_2,q_3)$ and $\vec{p}=(p_1,p_2,p_3)$, and the dynamics is given by the Hamiltonian energy function

$$H(q,p)=\frac{p^2}{2m}\,. \tag{3.1}$$

In the quantum system the conjugate dynamical variables are

$$\omega=(\vec{q},\vec{p})$$

and (3.2)

$$\pi=(\vec{\pi}^q,\vec{\pi}^p)\,.$$

Following the prescription (2.16) for the time development operator, we find

$$\mathcal{H}=+\frac{1}{m}\,\vec{p}\cdot\vec{\pi}^q\,. \tag{3.3}$$

Notice that this operator is linear in the conjugate momenta π. The superselection principle specifies that the ω^μ are superselecting operators whereas the π^μ are unobservable operators. The equations of motion for the dynamical variables in the Heisenberg picture are

$$\begin{aligned}&\dot{p}_j(t)=0\,,\quad \dot{q}_j(t)=\frac{1}{m}\,p_j(t)\,,\\&\dot{\pi}^q_j(t)=0\,,\quad \dot{\pi}^p_j(t)=-\frac{1}{m}\,\pi^q_j(t)\,,\end{aligned}\tag{3.4}$$

with corresponding solutions, in terms of the operators at an initial time

$$\begin{aligned}&p_j(t)=p_j(0)\,,\quad q_j(t)=q_j(0)+\frac{1}{m}\,p_j(0)t\,,\\&\pi^q_j(t)=\pi^q_j(0)\,,\quad \pi^p_j(t)=\pi^p_j(0)-\frac{1}{m}\,\pi^q_j(0)t\,.\end{aligned}\tag{3.5}$$

The first pair of equations mimic for the observable operators the classical solutions. The equations and solutions for the unobservables cannot be physically checked, but they are necessary to ensure the internal consistency of the quantum theory.

The solutions (3.5) are operator solutions. We must also specify the state of the system. Classically the state chosen is a pure state: p_j and q_j have exact values initially. Correspondingly in this case we can choose the state representing the system to be a pure state—i.e., an eigenstate of the complete commuting set of observable operators q_i and p_i, of the form

$$|\Psi\rangle=|q_0,p_0\rangle\,, \tag{3.6}$$

where

$$p_j|q_0,p_0\rangle = p_{j_0}|q_0,p_0\rangle$$

and

$$q_j|q_0,p_0\rangle = q_{j_0}|q_0,p_0\rangle .$$

Then at a later time t the values of the operator solutions (3.5) are found by taking the expectation values, viz.

$$\langle p_j(t)\rangle_\Psi = p_{j_0}\,, \quad \langle q_j(t)\rangle_\Psi = q_{j_0} + \frac{1}{m}\,p_{j_0}t\,. \tag{3.7}$$

So if initially the particle has coordinates p_{j_0}, q_{j_0}, then at time t it has coordinates p_{j_0}, $q_{j_0}+(1/m)p_{j_0}t$, just exactly the classical solution.

If we had worked in the Schrödinger picture, the state of the system would be seen to develop from the state $|\Psi,0\rangle \equiv |\Psi\rangle$, defined by (3.6), to the state $|\Psi,t\rangle$ on which q_j and p_j would yield the values (3.7). Clearly $|\Psi,0\rangle$ and $|\Psi,t\rangle$ belong to different superselected sectors of the space of state vectors.

For the classical particle moving in a conservative force field the Hamiltonian energy function is

$$H(q,p) = \frac{p^2}{2m} + V(q)\,, \tag{3.8}$$

where

$$\vec{\mathrm{F}} = -\frac{\partial}{\partial \vec{\mathrm{q}}}\,V(q)\,.$$

Then the corresponding Hamiltonian operator is

$$\mathcal{H} = +\frac{1}{m}\,\vec{\mathrm{p}}\cdot\vec{\pi}^{q} - \frac{\partial V(q)}{\partial \vec{\mathrm{q}}}\cdot\vec{\pi}^{p}\,. \tag{3.9}$$

The corresponding Heisenberg equations of motion are

$$\begin{aligned} &\dot{p}_j(t) = -\frac{\partial}{\partial q_j}\,V(q) = F_j\,, \quad \dot{q}_j(t) = \frac{1}{m}\,p_j(t)\,,\\ &\dot{\pi}^q_j(t) = \frac{\partial^2 V(q)}{\partial q_j \partial q_i}\,\pi^p_i(t)\,, \quad \dot{\pi}^p_j(t) = -\frac{1}{m}\,\pi^q_j(t)\,, \end{aligned} \tag{3.10}$$

again giving us the classical equations of motion for the observables.

For the simple harmonic oscillator the potential energy function is $V(q) = (1/2)m\omega^2q^2$, and in this case the solutions to the equations are

$$\begin{aligned} q_j(t) &= q_j(0)\cos\omega t + \frac{1}{m\omega}\,p_j(0)\sin\omega t\,,\\ p_j(t) &= p_j(0)\cos\omega t - m\omega q_j(0)\sin\omega t\,,\\ \pi^p_j(t) &= \pi^p_j(0)\cos\omega t - \frac{1}{m\omega}\,\pi^q_j(0)\sin\omega t\,,\\ \pi^q_j(t) &= \pi^q_j(0)\cos\omega t + m\omega\pi^p_j(0)\sin\omega t\,. \end{aligned} \tag{3.11}$$

Applying these operator solutions to the state vector, for example that given by (3.6), gives us back the correct classical solutions for the observable sector of the theory.

Thus, we have explicitly demonstrated some, admittedly simple, examples of quantum theories, of which the observable sector is exactly a well-known classical system. The superselection principle ensures that superposed states are not allowed as pure states of the system, the latter being given by states with sharp values of the complete set of (commuting) observables as defined by (2.10) and (2.11).

IV. THE SCHRÖDINGER PICTURE

In the previous sections we have discussed a method whereby a classical-mechanical system, whose dynamics can be written in the Hamiltonian formulation, could be embedded as the observable sector of a quantum-mechanical theory. Because the dynamics of a quantum theory in the Heisenberg picture more closely resembles that of the classical Hamiltonian formulation, we chose to work in that picture. One is more accustomed, on the other hand, to think in terms of the Schrödinger picture when discussing quantum-mechanics problems. For this reason we will now include a discussion of the Schrödinger picture time development.

In discussing the Schrödinger picture, one can either choose to work in a "wave mechanics" formulation, or in terms of the vector space method. Wave mechanics treats the q,p representation of the state of a system. Then the operators conjugate to q and p can be represented by

$$\pi^q_j = -i\,\frac{\partial}{\partial q_j}\,, \quad \pi^p_j = -i\,\frac{\partial}{\partial p_j}\,. \tag{4.1}$$

The system is described by a wave function which we denote in the following ways:

$$\psi(\omega) = \psi(q,p) \equiv \psi(q,p,0)\,. \tag{4.2}$$

With the Hamiltonian operator given by Eq. (2.16),

$$\mathcal{H} = i\left[\frac{\partial H(q,p)}{\partial q_j}\,\frac{\partial}{\partial p_j} - \frac{\partial H(q,p)}{\partial p_j}\,\frac{\partial}{\partial q_j}\right]\,,$$

the Schrödinger time development of the wave function is

$$\psi(q,p,0) \to \psi(q,p,t) = e^{-i\mathcal{H}t}\psi(q,p,0)\,.$$

The Schrödinger equation is, then,

$$i\,\frac{d}{dt}\,\psi(q,p,t) = \mathcal{H}\psi(q,p,t)\,. \tag{4.3}$$

In quantum mechanics it is usual to expand $\psi(q,p,t)$ in terms of the eigenstates of $\mathcal{H}$, thus simplifying the problem. In this case, $\mathcal{H}$ is not an observable, so we choose not to follow this procedure. Equation (4.3) is a first-order partial differential equation, and so its solution follows quite simply.[5] We define

$$b_i(q,p) = -\frac{\partial H(q,p)}{\partial p_i}\ ,\quad d_i(q,p) = \frac{\partial H(q,p)}{\partial q_i}\ . \qquad (4.4)$$

Then the equation takes the following form:

$$\left[\frac{\partial}{\partial t} - b_i(q,p)\frac{\partial}{\partial q_i} - d_i(q,p)\frac{\partial}{\partial p_i}\right]\psi(q,p,t) = 0\ , \qquad (4.5)$$

a form reminiscent of the renormalization group equations in quantum field theory.[6] To solve this equation we introduce the functions $\overline{q}(q,p,t)$ and $\overline{p}(q,p,t)$, where

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{q}_j(q,p,t) = b_j(\overline{q},\overline{p})\ ,\quad \overline{q}_j(q,p,0) = q_j\ ,\\ &\frac{\partial}{\partial t}\overline{p}_j(q,p,t) = d_j(\overline{q},\overline{p})\ ,\quad \overline{p}_j(q,p,0) = p_j\ .\end{aligned} \qquad (4.6)$$

Then the solution to Eq. (4.5) is

$$\psi(q,p,t) \equiv \psi(\overline{q}(q,p,t),\overline{p}(q,p,t))\ , \qquad (4.7)$$

i.e., the Schrödinger equation puts no restrictions on the allowed initial form of the wave function. If at time $t=0$ the wave function is $\psi(q,p)$, then at time t the wave function of the system is $\psi(\overline{q},\overline{p})$. The equations determining $\overline{q}$ and $\overline{p}$ are, using (4.4) and (4.6), very similar to the classical equations of motion, apart from a minus sign in each of the equations.

It is relatively easy to understand why this result should hold. We are after all examining a problem of classical physics. The wave function $\psi(q,p,0)$ is interpreted as the probability amplitude that the system is in an eigenstate of the observables corresponding to the eigenvalues q and p.

From equations (4.6) and (4.4) it is clear that

$$\begin{aligned}&\overline{q}_j(q,p,t) = q_j(-t)\ ,\\ &\overline{p}_j(q,p,t) = p_j(-t)\ ,\end{aligned}$$

where $q_j(-t)$ and $p_j(-t)$ are the solutions of the classical problem at time $-t$. The solution to the Schrödinger equation can be written as

$$\psi(q,p,t) = \psi(q(-t),p(-t),0)\ ,$$

i.e., the probability amplitude that the system is described by the phase-space point (q,p) at time t equals the probability amplitude that at time 0 the system was described by the phase-space point $(q(-t),p(-t))$. Of course classically $(q(-t),p(-t))$ at time 0 are the initial conditions for the system to be described by (q,p) at time t. Thus the Schrödinger picture also leads quite simply to the classical interpretation of the quantum theory we have constructed earlier.

So far we have seen how to embed a classical system in a larger quantum-mechanical structure. We outlined the procedure in the Heisenberg picture initially. In this section we have seen how the Schrödinger time development also leads us to the same description, but along a different path. Our discussions so far were restricted to isolated purely classical systems. In the next sections we will examine a particular type of interaction, namely the interaction of a classical and a truly quantum system. Our purpose is to see if such an interaction can lead to a description of a measurement process.

V. PROPOSAL OF A MODEL FOR THE QUANTUM MEASUREMENT PROCESS

In the preceding sections of this paper we have discussed an unconventional treatment of classical systems. We wish now to discuss one possible application of this model. This application is in the realm of measurement theory, in particular the measurement of quantum systems.

In a measurement problem one has two separate systems, one of which is the apparatus, or measurer, while the other is the system to be examined. It is our premise that a piece of apparatus can be described by classical physics. We propose to treat the apparatus as a truly classical system. Thus, our description of the measurement of properties of quantum systems involves the coupling of a classical apparatus to a quantum system.

The purpose of dealing with the measurement process is to reproduce theoretically what is experimentally seen, if that is possible: The interaction between apparatus and quantum system results in a change in the "settings," or state, of the apparatus from which knowledge of the state of the quantum system is deduced.

There are many levels at which this problem can be addressed. At the first level we consider only closed systems, about which we have maximum knowledge allowed by the theory, in principle. At such a level, the classical apparatus might be assumed to be a closed mechanical system of the type discussed in Sec. II. The quantum system would be described by elementary quantum mechanics, even if it is not an elementary system. Thus, the states of the quantum system would be pure states and their development in time would be by a unitary transformation.

At a higher level, one can take account of unknown external influences on the actual (physical) quantum system by admitting one's ignorance as to the precise form of the Hamiltonian. One way to proceed, then, is to consider a probability distribution over a set of Hamiltonians which differ from each other by different values for the unknown external parameters. It is then more convenient to use density matrices rather than pure states in the description of the quantum system. One result of this procedure is that the time development is no longer described by a unitary transformation on the space of density matrices.[7]

In this paper we shall restrict our attention to the first level. We are more interested in problems of principle than in problems of complexity. If our model yields interesting results at the lowest level, it would then be imperative to extend it to higher levels also. On the other hand, if our model does not yield interesting results at the lower level, and by this we mean that it will not be possible to interpret the interactions between the classical apparatus and the quantum system as leading to a measurement, then we should abandon the approach. Extension of the model, in that case, to the higher levels would be most unlikely to alter the result.

When the apparatus is coupled to the quantum system, how do we know if a measurement results? In our model, since we treat both the quantum system and the quantum-enlarged apparatus in their idealistic form, it is clear that the coupling will result in a correlated total system. To begin with we only require the following: A measurement is achieved if *unambiguous* information concerning the values of certain variables of the quantum system being examined can be "stored" in the variables of the classical apparatus. We shall return to the problem of the transfer of this information to an observer at another time.

Our proposal is to treat the apparatus as a classical system. Is this justifiable? One can think of experiments where the apparatus might not be a classical system. Is, then, our model not applicable to such cases? In answer we note that even if the apparatus is not strictly classical, all we ever use of its dynamical variables is a commuting set. We would label these the observables, and any other variables unobservables. Then, in a sense, the quantum-enlarged model considered earlier would now be the conventional description of the apparatus. We see then that there are two levels at which a system might be considered classical:

(1) a truly classical system

(2) a system with a commuting set of observables.

In Sec. VI we will discuss how one should proceed to couple together the quantum-enlarged classical system, and the truly quantum system, so as to achieve an adequate description of a measurement.

VI. INTERACTION BETWEEN CLASSICAL AND QUANTUM SYSTEMS: GENERAL CONSIDERATIONS

In this section we examine the construction of interactions between classical and quantum systems. Following the spirit of this paper we restrict our attention to general considerations. We concentrate on the general features which characterize the interacting theories and return to discuss the interactions in more detail in a subsequent paper.[8]

As emphasized in Sec. V, the sole purpose of embedding a classical system into a larger quantum framework is to allow a classical system to interact with a truly quantum sustem for the purpose of achieving a measurement. Thus we must examine what interactions can be constructed between the quantum-enlarged classical system and the quantum system under examination. We must examine how, if at all, such interactions affect the classical nature of the apparatus.

Let us first introduce some notation. We denote the algebra of dynamical variables of the quantum system by $\{\xi\}$. The quantum system is fully specified when the energy operator, the state vector and the commutation rules of the algebra are known. We write the Hamiltonian of the isolated quantum system as $X(\eta)$, where $\{\eta\}$ is some subset of the quantum variables. The Hamiltonian operator for the uncoupled systems is, then,

$$\mathcal{H}_0=\frac{\partial H(\omega)}{\partial\omega^\mu}\,\epsilon^{\nu\mu}\pi^\nu+X(\eta)\,. \tag{6.1}$$

To describe the measurement process within the overall dynamical framework we generate, in the Hamiltonian operator for the total system, a coupling between the apparatus and the quantum system. In this way, the measurement process will be seen to occur as the natural time development of the system. The Hamiltonian will then be

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{\text{int}}\,, \tag{6.2}$$

where

$$\mathcal{H}_{\text{int}}=\Phi(\omega,\pi;\xi';t)\,. \tag{6.3}$$

Here $\{\xi'\}$ is some subset of the quantum variables, and we include an explicit time dependence in the interaction term to allow for more general couplings. The effect of the interaction (6.3) will be felt by both apparatus and quantum system, and in the case of the apparatus it may even be sufficient to destroy its classical nature.

It is necessary now to understand exactly what is the classical nature of the apparatus. What properties of the apparatus ensure that it is classical? In fact there are two properties which satisfy this requirement. The first is a statement about classical observables: The set of classical observables must form a commuting set. The second property is a statement about the classical state. To discuss this property it is convenient to make use of the Schrödinger picture. If the state of the (quantum-enlarged) classical system is chosen initially to be an eigenstate of the observables ω_{op},

$$|\psi, 0\rangle = |\omega_0\rangle \tag{6.4}$$

then for all times $t > 0$, the state vector $|\psi, t\rangle$ will also be an eigenstate of ω_{op}. This property follows directly from the discussion in Sec. IV. We may phrase this another way, using the fact that $\omega_{op}{}^{\mu}$ are superselecting operators: If the state of the system is initially chosen to be pure, then at all future times it is also a pure state.

The statement of this latter property in the Heisenberg picture is as follows: If the state of the classical system, denoted by $|\psi\rangle$, is chosen to be an eigenstate of $\omega_{op}(0)$, then it is also an eigenstate of $\omega_{op}(t)$.

If we allow interactions of the very general form (6.2) between classical and quantum systems, then it is relatively straightforward to see that the property of the classical state, which we have just introduced, will be destroyed by almost all such interactions. The reason for this result goes as follows: Let us denote the initial state of the apparatus by $|\psi\rangle$ and the initial state of the quantum system by $|\phi\rangle$. Then the initial state of the total system (apparatus plus quantum system) is

$$|\Psi, 0\rangle = |\psi\rangle \otimes |\phi\rangle . \tag{6.5}$$

The choice (6.4) is made, as usual, for $|\psi\rangle$. As a direct result of the interaction term (6.3), we can no longer claim that $|\Psi, t\rangle$ is also an eigenstate of ω_{op}. In fact the only cases when this would occur are if the interaction has no effect whatsoever on the apparatus, or if the quantity being measured is a true attribute of the quantum system. The first case is clearly uninteresting for our present purposes. In the second case, by attribute of the system we mean that $|\phi\rangle$ already is an eigenstate of $\mathcal{H}_{int}$, i.e., we can predict with certainty the result of each experiment. However, in all cases except these two special cases, $|\Psi, t\rangle$ is no longer an eigenstate of ω_{op}. In fact, this is merely another manifestation of the nonseparability of quantum systems.[1]

To the extent that this property of the state of a classical system is not retained, it may be said that the classical system does not remain purely classical if it is allowed to interact with a purely quantum system.

But this is just one of the classical properties. Let us now examine how interactions of the form (6.2) affect the classical nature of the classical observables. To discuss this property it is convenient to use the Heisenberg picture, as in that formulation it is the dynamical variables which develop in time. The classical nature of the apparatus observables is characterized by both of the following properties:

(i) $\omega^{\mu}(t)$ are observable for all times t

(ii) $\omega^{\mu}(t)$ and $\omega^{\nu}(t')$ are compatible operators for all times t and t'.

The first property tells us that the "trajectories" of the apparatus observables are observable for all times. The second tells us that we can measure the different trajectories without disturbing the measurable aspects of the system. These two properties are not independent of one another. The first requires that the commutator

$$[\omega^{\mu}(t), \omega^{\nu}(0)] \tag{6.6}$$

vanish for all times t, and for all μ, ν. To satisfy (ii) the commutator

$$[\omega^{\mu}(t), \omega^{\nu}(t')]$$

must vanish, for which it suffices to consider commutators of the form (6.6) because

$$[\omega^{\mu}(t), \omega^{\nu}(t')] = e^{i\mathcal{H}t'}[\omega^{\mu}(t-t'), \omega^{\nu}(0)]e^{-i\mathcal{H}t'} . \tag{6.7}$$

We note that for the uncoupled classical system properties (i) and (ii) are automatically satisfied because the Hamiltonian (2.16) is at most linear in the unobservables $\pi_{op}{}^{\mu}$.

We now turn to the general Hamiltonian (6.2) to see whether or not these equivalent properties of the apparatus observables are retained in the time development of the interacting system. Clearly, if the coupling function Φ is quadratic (or higher) in the unobservables $\pi_{op}{}^{\mu}$ the apparatus observables will no longer be characterized by (i) and (ii) after the interaction has taken place. On the other hand, if the coupling function is linear in $\pi_{op}{}^{\mu}$ this result does not follow. In that case it may occur that properties (i) and (ii) are retained even in the presence of some interactions of the form (6.2).

From the foregoing discussion it is clear that couplings between a classical system and a quantum system could destroy the nature of the classical system. Obviously, the use of a classical apparatus makes sense only if the apparatus is also to be classical after the interaction has taken place. However, since the "classical state" cannot be retained, except for interactions which are not general enough for our purposes, we propose that we restrict our attention to interactions which preserve, in some sense, the classical nature of the apparatus observables.

This is a weaker requirement than the possibility of requiring the classical nature of the apparatus state to be preserved. It is also the minimum requirement, since if it is not satisfied, it would not be possible to consider the apparatus as being classical, in any sense, after the interaction has occurred.

We formulate this requirement as a principle of integrity. It is the integrity of the classical observables which is to be preserved. In so far as comparison with experiment is concerned, we may formulate a weak form of the principle. This merely requires that after the interaction has ceased, the apparatus observables should retain their classical integrity. While the interaction is taking place, no such restriction is enforced. This form of the principle is most useful when discussing interactions which are explicitly time dependent.

We may also formulate a strong form of the principle, according to which the interactions should preserve the classical integrity of the apparatus observables at all times. This more restrictive form is useful when we are dealing with time independent interactions.

Whichever form of the principle of integrity that we choose to work with, we restrict our attention to interaction terms of the form

$$\mathcal{H}_{\text{int}} = \phi^{\mu}(\omega;\eta';t)\pi^{\mu} + h(\omega;\epsilon;t)\,. \tag{6.8}$$

Here $\{\eta'\}$ and $\{\epsilon\}$ are subsets of the quantum variables. We again include the possibility of an explicit time dependence.[9] Such interactions (of an impulsive type), where the time dependence is of the δ-function type, have been considered previously.[3]

The form (6.8) for $\mathcal{H}_{\text{int}}$ is not, however, sufficient to guarantee that the apparatus observables retain their classical integrity. Both the primary coupling functions ϕ^{μ} and the secondary coupling function h depend on unspecified quantum variables. What we need to do is to find what *further* restrictions, if any, on the functional form of these coupling functions can be deduced by requiring the principle of integrity to be satisfied. Or, if that fails, we need to derive criteria which can be used to check different models. Clearly it is not *a priori* obvious that any interactions would preserve the classical integrity of the apparatus observables.

We have derived, in the case of the strong form of the principle of integrity, a set of criteria which can be used to check whether or not a given interaction will satisfy the principle. This analysis, along with its illustration in the case of a simple example will be presented in a subsequent paper, II in the present series. There we will see that it is possible to have an interaction between a classical and a quantum system which satisfies the principle of integrity. The example can be viewed as a variant of the Stern-Gerlach experiment, where the interaction is of an artificial form.

VII. CONCLUDING REMARKS

The purpose of this paper was to introduce a new approach to the description of the quantum measurement process. We have concentrated mainly on our major tool: the description of a classical object in a quantum-mechanical fashion. We have shown how to embed a classical Hamiltonian system within a quantum framework with twice the number of degrees of freedom. The concepts of superselecting operators and unobservable dynamical variables have been essential in setting up the formalism.

In Secs. V and VI we have proposed to use this tool explicitly in the measurement process. We treat the apparatus as a classical system and couple it to the quantum system under investigation. The coupling is achieved by making use of the model introduced in Sec. II to describe the classical apparatus.

We have considered the question as to what interactions should be allowed. Use of a classical apparatus leads us to require that the apparatus should remain classical. Nevertheless, we saw that the state of the system does not remain classical in nature. We formulated a principle of integrity for use at this juncture. It is the integrity of the classical observables which is to be retained when interactions occur. The principle is formulated in both weak and strong forms:

(i) weak form. The classical observables of the apparatus must retain their classical integrity after the coupling between the apparatus and quantum system has ceased;

(ii) strong form. The classical observables of the apparatus must retain their classical integrity at all times, even when interactions with a quantum system are allowed.

Requiring that the interactions satisfy this principle is the weakest requirement we can impose if we wish the apparatus to be "classical," in any sense, after interacting with the quantum system.

We note, in passing, that the principle has been formulated in the Heisenberg picture. We defer to a subsequent paper an analysis, also in the Heisenberg picture, of the restrictions such a principle places on the allowed interactions, illustrating by means of a simple example.

We must emphasize that in our work so far we have restricted our attention to idealistic realizations, where the quantum systems are closed systems, and maximum information about them is known. The actual measurement problem involves the recording of data, and, thus, the general problem of irreversible processes. In the treatment of irreversible processes one is led to a discussion of thermodynamic systems. If our approach yields interesting conclusions at its present restricted level, it will become necessary to extend it to a more realistic level.

There exists an interesting connection between the ideas we have presented here and recent work in nonequilibrium thermodynamics.[10,11] A crucial problem in nonequilibrium thermodynamics is the definition of a noequilibrium entropy in terms of the concepts of classical mechanics and the explanation of its monotonic increase on the basis of Hamiltonian dynamics. Poincaré proved, a long time ago,[12] that there cannot exist a phase function which is of definite sign and which increases monotonically to a maximum under Hamiltonian evolution. Thus the second law of thermodynamics must have its dynamical counterpart in the study of dynamical variables other than phase functions. This problem has been investigated recently by Misra,[11] whose analysis suggests quite strongly that if a *classical* system is to display a thermodynamic behavior its algebra of dynamical variables must be noncommutative. Our quantum-enlarged classical systems automatically provide a rich framework within which a true entropy function may be constructed. If such ideas prove useful, they will answer in part some of the questions raised in the Introduction.

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[1]See for example the excellent book by B. d'Espagnat, and the many references contained therein: *Conceptual Foundations of Quantum Mechanics*, 2nd ed. (Benjamin, Reading, 1976).

[2]This same process of destruction (or modification) of correlations may also underlie the irreversible process involved in the indelible actions produced in measurement. But a discussion of such a topic is beyond the scope of this paper.

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[5]See for example E. T. Copson, *Partial Differential Equations* (Cambridge Univ. Press, Cambridge, 1975).

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[9]In such cases the time development factors $\exp(\pm i \mathcal{H} t)$ should be replaced by the time-ordered quantities $T\{\exp[\pm i \int_0^t \mathcal{H}(s)ds]\}$. By means of a time-dependent interaction it would be possible to artificially start and stop the interaction between the apparatus and quantum system.

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