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Form of relativistic dynamics with world lines

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In any Hamiltonian relativistic theory there are ten generators of the Poincaré group which are realized canonically. The dynamical evolution is described by a Hamiltonian which is one of the ten generators in Dirac's generator formalism. The requirement that the canonical transformations reproduce the geometrical transformation of world points generates the world-line conditions. The Dirac identification of the Hamiltonian and the world-line conditions together lead to the no-interaction theorem. Interacting relativistic theories with world-line conditions should go beyond the Dirac theory and have *eleven* generators. In this paper we present a constraint dynamics formalism which describes an eleven-generator theory of N interacting particles using $8(N+1)$ variables with suitable constraints. The $(N+1)$th pair of four-vectors is associated with the uniform motion of a center which coincides with the center of energy for free particles. In such theories dynamics and kinematics cannot be separated out in a simple fashion.

I. INTRODUCTION

The problem of describing relativistic interacting point particles in the Hamiltonian formalism of dynamics has been with us for a long time. More than three decades ago Dirac[1] proposed a very general framework—the so-called generator formalism—for relativistic Hamiltonian dynamics. The basic idea is that one should have a phase-space structure on which the group of inhomogeneous Lorentz transformations—the Poincaré group $\mathcal{P}$—is realized by canonical transformations; and this requires that we have ten generators that give us a Poisson-bracket realization of the Lie algebra of $\mathcal{P}$. In Dirac's work the question of dynamical evolution within one inertial frame was subsumed under the more general question of representing a change of inertial frame corresponding to any element of $\mathcal{P}$; so "equations of motion" were particular cases of more general equations describing the effect of infinitesimal Poincaré transformations. Thus both "Hamiltonian" and "interaction" were contained among the ten generators of $\mathcal{P}$.

Soon after Dirac's work, Thomas,[2] Bakamjian,[3] and Foldy[4] succeeded in constructing theories of interacting relativistic point particles in the generator formalism and their work was extended to quantum theory by Macfarlane, Jordan, and Sudarshan[5] and by Mukunda.[6] Thomas and Bakamjian gave up the concept of the objective reality of particle world lines. The idea here is that when the observations in two inertial frames θ and θ' are related by the canonical transformation representing the appropriate element of $\mathcal{P}$, then in any particular state of motion, θ' and θ do in fact "see" the same set of world lines in spacetime, i.e., the canonical and the geometrical (Lorentz) transformations are compatible when applied to the particle positions. We shall refer to this as the requirement of invariant world lines. It had already been recognized much earlier by Pryce[7] that this requirement entailed conditions that definitely went beyond merely obeying the structure relations of $\mathcal{P}$ in a canonical framework. These world-line conditions (WLC's) were put into a convenient phase-space form in Poisson-bracket notation by Currie, Jordan, and Sudarshan.[8,9] They succeeded in proving that, within the Dirac generator formalism, relativistic invariance, interaction, and invariant world lines were mutually incompatible for a system of point particles. This is the no-interaction theorem.

In response to the challenge represented by this theorem, there have recently been several papers that succeed in constructing models of relativistic interacting point particles within a constrained Hamiltonian formalism. Constrained Hamiltonian dynamics, originally devised by Dirac[10] to handle singular Lagrangian systems, can be made the basis of a self-contained independently existing formalism for construction of theories. Its property of capturing the virtues of both the Lagrangian and conventional Hamiltonian forms of dynamics, and yet being more flexible than either, has been specially stressed by Komar.[11] The characteristic feature of the constraint formalism, if it does not arise from an underlying Lagrangian, is that the identification of the mathematical variables with physical quantities is delayed until the stage at which all needed constraint conditions have been given.

At first it appeared that the assumption underlying the no-interaction theorem that had apparently been given up in these models with interaction was the existence of invariant world lines. However, a more careful examination presented elsewhere has shown that the matter is much deeper than that: One has gone beyond the Dirac generator formalism and separated the question of dynamical evolution within one inertial frame from the action of the Poincaré group.[14] A Hamiltonian relativistic system in this framework requires eleven generators to define it; one to provide "equations of dynamical evolution" in each frame, and ten to build a canonical realization of $\mathcal{P}$.

This implies sacrificing the hitherto implicit sharp separation between kinematics and dynamics that one is accustomed to in Lagrangian dynamics and that was assumed in Dirac's work. While one may be reluctant to take such a step, it certainly makes it now possible to have relativistic invariance, interaction, and invariant world lines, all at the same time. We shall therefore take this step.

For a system of point particles, such a scheme can be implemented in more than one way—this again points to the great flexibility of the constraint formalism. One expects that for an isolated dynamical system there is some hypothetical "center" in it—not necessarily coinciding with any constituent of it—that, as a result of Poincaré invariance, travels uniformly in a straight line, in any state of motion of the system. The difficulties in identifying such a center were discussed long ago by Pryce.[7] In our extended relativistic Hamiltonian framework we now have two possibilities: (1) We may introduce just the right number of variables, indexed by particle number, so that when all the constraints have been imposed we have the right number of degrees of freedom, and we permit the "center" to emerge from the overall relativistic description; (2) alternatively we may deliberately introduce one set of phase-space variables more than the number of particles would warrant, intended to describe a center, and rely on the constraint mechanism to reduce the extra variables. The latter method has been advocated by Rohrlich[13] in several papers.

Even though both these approaches succeed in producing models of interacting particles with invariant world lines, they are not quite identical in structure. An analysis of the models based on approach (1) has been presented elsewhere.[14] Here we study approach (2), bringing out clearly and consistently its departure from the Dirac framework, and the existence of invariant world lines.

II. THE RELATIVISTIC N-PARTICLE SYSTEM

Our intention is to describe a system of N interacting point particles. The final form of the theory should then involve exactly $6N$ independent variables with suitable bracket relationships.

We begin, however, with an artificially expanded $(8N+8)$-dimensional phase space Γ' with independent "four-vector" variables Q_μ, P_μ, $\xi_{a\mu}$, $\eta_{a\mu}$, $a=1,2,\ldots,N$. The only nonzero Poisson brackets (PB's) are taken to be

$$\{Q_\mu, P_\nu\} = g_{\mu\nu}, \quad \{\xi_{a\mu}, \eta_{b\nu}\} = \delta_{ab} g_{\mu\nu}. \tag{1}$$

The intention is that Q should be the space-time position of a center, P its constant four-momentum, ξ_a the position of particle a relative to the center in the frame in which the center is at rest, and η_a the momentum of particle a in that frame. These intentions will be realized, and the system itself acquires definition, through a series of constraints.

On Γ' the Poincaré group $\mathcal{P}$ can be canonically realized in this way:

$$(\Lambda, a)\in\mathcal{P}: Q_\mu \to \Lambda_\mu{}^\nu Q_\nu + a_\mu, \quad \xi_{a\mu} \to \Lambda_\mu{}^\nu \xi_{a\nu},$$
$$P_\mu \to \Lambda_\mu{}^\nu P_\nu, \quad \eta_{a\mu} \to \Lambda_\mu{}^\nu \eta_{a\nu}. \tag{2}$$

Note that Q alone responds to a space-time translation. The ten infinitesimal generators of the above canonical transformations, providing a PB realization of the Lie algebra of $\mathcal{P}$, are

$$J_{\mu\nu} = Q_\mu P_\nu - Q_\nu P_\mu + \sum_{a=1}^{N} (\xi_{a\mu}\eta_{a\nu} - \xi_{a\nu}\eta_{a\mu}) \tag{3}$$

for the homogeneous Lorentz group, and P_μ itself for the translations.

Let us immediately reduce the number of variables from $8N+8$ to $6N+8$ by imposing the $2N$ independent second-class constraints

$$P\cdot\xi_a \approx 0, \quad P\cdot\eta_a \approx 0, \quad a=1,\ldots,N. \tag{4}$$

The nontrivial parts of ξ, η therefore transform as vectors under the little group of $\mathcal{P}$ with respect to P_μ. These conditions define a constraint surface Γ in Γ'. It is clear that the mappings (2) carry Γ into itself. We can explicitly get rid of the extra variables by passing from the PB's (1) to the Dirac brackets[10] (DB's) corresponding to the constraints (4). Since we shall hereafter not have to make any reference to Γ' or the PB's (1), we shall use the symbol $\{\ ,\ \}$ for the DB's obtained at this stage. The nonvanishing DB's are

$$\{Q_\mu, Q_\nu\} = -\frac{1}{P^2}\sum_a (\xi_{a\mu}\eta_{a\nu} - \xi_{a\nu}\eta_{a\mu}), \quad \{Q_\mu, P_\nu\} = g_{\mu\nu},$$
$$\{Q_\mu, \xi_{a\nu}\} = -\xi_{a\mu}\frac{P_\nu}{P^2}, \quad \{Q_\mu, \eta_{a\nu}\} = -\eta_{a\mu}\frac{P_\nu}{P^2}, \tag{5}$$
$$\{\xi_{a\mu}, \eta_{b\nu}\} = \delta_{ab}\left(g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}\right).$$

From now on it is understood that the conditions (4) hold as identities. Based on these brackets (5), the ten variables $J_{\mu\nu}, P_\mu$ continue to yield a realization of the Lie algebra of $\mathcal{P}$; so a finite element (Λ, a) in $\mathcal{P}$ is realized by a transformation $R(\Lambda, a)$ on Γ which is canonical with respect to the brackets (5). The action of $R(\Lambda, a)$ on Q, P, ξ, η is exactly as before, namely, it is given by Eq. (2), even though these are not all independent variables.

At this point we introduce a pair of variables (q_a, p_a) for each value of a, with the intention that in the final theory these shall be interpreted as "four-position" and "four-momentum" of particle number a:

$$q_a = Q + \xi_a, \tag{6a}$$

$$p_a = \eta_a + (m_a^2 - \eta_a^2 + V_a)^{1/2} P/(P^2)^{1/2}, \quad a = 1, 2, \ldots, N. \tag{6b}$$

These variables do not have any special canonical bracket properties. The V_a are a set of "interaction potentials," to be further described shortly. To preserve covariance under $R(\Lambda, a)$, we only demand that each V_a be invariant under the action of $R(\Lambda, a)$; we also exclude any dependence of V_a on P for a reason to be explained later; thus each V_a can be any Lorentz-invariant function of ξ, η. Then each q_a transforms like Q, i.e., like space-time position, and each p_a like P, under $\mathcal{P}$.

At this stage, since Γ has dimension $6N+8$, we are in need of eight more independent constraints. Only then can we make contact with a system of *N particles*. The definition of p_a, together with the notion that P is the total four-momentum, suggests immediately four independent constraints we may impose:

$$C = \sum_a \eta_a \approx 0, \qquad D = (P^2)^{1/2} - \sum_a (m_a^2 - \eta_a^2 + V_a)^{1/2} \approx 0. \tag{7}$$

Note that the component of the C equation in the direction of P_μ vanishes by virtue of (4) so that there are only three independent C constraints and one D constraint. Then it will indeed be true that the p_a all add up to P. Now the theory of constrained systems reminds us that if C, D form a first-class system, one naturally has room for the introduction of four more constraints as foils to these. Then our counting of degrees of freedom would be just right. The three independent functions in C do have vanishing brackets with one another. Let us now demand that the brackets $\{D, C\}$ also vanish. The simplest way to achieve this is to make each $\{C, V_a\}$ vanish. From the brackets (5) we see that this can be arranged if each V_a depends only on the *differences* of ξ's, and on the η's, in a Lorentz-invariant way. *This restriction on V_a will now be assumed.*

The region in Γ where the four independent constraints (7) hold is a region Σ of dimension $6N+4$. of C and D, $R(\Lambda, a)$ maps Σ onto itself: namely, we have

$$\{J_{\mu\nu} \text{ or } P_\mu,\ C \text{ or } D\} \approx 0. \tag{8}$$

Let us now consider the canonical transformations generated by C, D. Because of the first-class property, (i) these transformations map Σ onto itself, (ii) they form a four-parameter group. In the present case, this group happens to be Abelian since the brackets among C and D vanish identically, i.e., strongly. Starting with some point in Σ, we can apply all the canonical transformations in this Abelian group and thus build up the "orbit" of that point under this group: This will be a *region of dimension four,* contained entirely within Σ, and we will call it a "sheet." We can display a sheet S graphically in this way: Let X collectively symbolize all the coordinates Q, P, ξ_a, η_a of a point on Σ. Set up the differential equation

$$\frac{dX(\sigma)}{d\sigma} = \{X(\sigma), D\} w + \{X(\sigma), C\} v, \qquad P \cdot v = 0, \quad X(0) = X. \tag{9}$$

If we imagine solving these equations for all possible choices of the four free multipliers w, v and collect together all the points of Σ that have been reached in this way from the given X, we get the sheet S containing X. [It is of course understood that the arguments of D, C in Eq. (9) are the variables $X(\sigma)$ themselves.]

Each sheet S is determined by any one of the points it contains, and Σ is seen to be the union of disjoint sheets. It follows that the sheets form a $6N$-parameter family. Because of (8) one also sees that the transformation $R(\Lambda, a)$ maps each sheet S entirely onto another sheet S'. One can now consider whether it is physically reasonable to imagine that each sheet S, as a whole, corresponds to one state of motion of an N-particle system. If it were so, then one could say that the change of inertial frame $\theta \to \theta' = (\Lambda, a)\theta$ is implemented by the action of $R(\Lambda, a)$ on the sheets. However, we discard this possible interpretation for this reason: If a sheet S be chosen, and *all* points in it be used in order to reconstruct a set of world lines in space-time, relative to an inertial frame θ, we will *not* end up with a line[14] *but a whole region of space-time*[15] *for each particle.* S is of dimension four, so as a point X varies over it, the variables $q_a^\mu(X)$ for each a form a four-parametric set of points in space-time. Our aim is to define a "state of motion" for the system in such a way that it will let us construct in an unambiguous way, in each frame θ, a set of N world lines. Once such a definition has been made, we can then develop a WLC to test if these are invariant world lines or not.

It is clear that this can be achieved by picking, on each sheet S, a one-dimensional curve $\mathcal{C}$ in some way, and then parametrizing the points on $\mathcal{C}$ by an evolution parameter τ. To fix $\mathcal{C}$ on S requires three independent constraints χ_r, $r=1,2,3$, dependent only on the phase-space variables; and a fourth one χ_4 with explicit τ dependence can then be added on to parametrize $\mathcal{C}$:

$$\begin{aligned} &\chi_r(Q,P,\xi,\eta)\approx 0\,, \quad r=1,2,3 \\ &\chi_4(Q,P,\xi,\eta,\tau)\approx 0\,. \end{aligned} \tag{10}$$

For these constraints to do what we want them to, it is necessary that along with the C, D they form a second-class set. If for the four independent conditions contained in (7) we write B_r, we must ensure that

$$\det|\{\chi_r, B_s\}|\neq 0\,. \tag{11}$$

We can then define each curve $\mathcal{C}$ to represent one state of motion. The rest of the sheet S containing $\mathcal{C}$ can be discarded as being of no physical consequence. With this definition of the term state of motion, we can see that in every such state, in a frame θ, a definite set of N world lines may be drawn: Along a $\mathcal{C}$, the variables (6a) appear in the form $q_a^\mu(\tau)$.

Let the inverse to the matrix (11) be written as $(\mathcal{a}_{rs})$:

$$\mathcal{a}_{rs}\{\chi_s, B_{s'}\}=\delta_{rs'}\,. \tag{12}$$

If we can explicitly eliminate the constraints χ, B, we will then be left with exactly $6N$ independent variables. To this end we pass from the brackets (5) to the (final) Dirac brackets

$$\begin{aligned} \{f,g\}^* = \{f,g\} & \\ &-\mathcal{a}_{rs}(\{f,B_r\}\{\chi_s,g\}-\{f,\chi_s\}\{B_r,g\}) \\ &-\{f,B_r\}\mathcal{a}_{rs}\{\chi_s,\chi_{s'}\}\mathcal{a}_{r's'}\{B_{r'},g\}\,. \end{aligned} \tag{13}$$

We now note several important things: Because of (8), the $J_{\mu\nu}, P_\mu$ continue to furnish a realization of the Lie algebra of $\mathcal{P}$ if we compute their DB's $\{\cdot,\cdot\}^*$ rather than their brackets according to (5). They may therefore be seen to build up a realization of $\mathcal{P}$ by transformations $R^*(\Lambda,a)$ which are canonical with respect to the DB $\{\cdot,\cdot\}^*$. The relationship between $R(\Lambda,a)$ and $R^*(\Lambda,a)$, for the same (Λ,a) in $\mathcal{P}$, is this: $R^*(\Lambda,a)$ maps Σ onto Σ as $R(\Lambda,a)$ does; if $R(\Lambda,a)$ takes a sheet S into S', $R^*(\Lambda,a)$ also carries S to S'; beyond this, $R^*(\Lambda,a)$ will automatically carry the $\mathcal{C}$ determined on S by (10) onto the $\mathcal{C}'$ determined on S', while preserving the value of τ. This last characterization is meaningless for $R(\Lambda,a)$.

The DB (13) is a nondegenerate bracket for a generalized phase space of $6N$ dimensions. It is cumbersome to exhibit explicitly a set of $6N$ independent variables spanning this space, but there is no need to do so either; we can work with all the variables Q, P, ξ_a, and η_a, with q_a and p_a being *derived* quantities given by Eqs. (6). The constraints (4), (7), and (10) are understood to hold as identities, and these are consistent with the DB (13). From this point, we physically identify $R^*(\Lambda,a)$ as representing the change of inertial frame $\theta\to\theta'=(\Lambda,a)\theta$. It is a transformation canonical with respect to the final physical brackets (13).

Along the curve $\mathcal{C}$ on an S, the equation of motion (9) with unspecified multipliers w, v becomes more definite: We must assign to w and the v's such values as will maintain the constraints (10) all along $\mathcal{C}$. Since χ_4 alone carries explicit τ dependence we find

$$\begin{aligned} &\frac{d\chi_r}{d\tau}=0 \Rightarrow \frac{\partial\chi_r}{\partial\tau}+\{\chi_r,B_s\}v_s=0 \\ &\Rightarrow v_r=-\mathcal{a}_{rs}\frac{\partial\chi_s}{\partial\tau}=-\mathcal{a}_{r4}\frac{\partial\chi_4}{\partial\tau}\,. \end{aligned} \tag{14}$$

For a general dynamical variable f the equation of motion along $\mathcal{C}$ would be

$$\begin{aligned} \frac{df}{d\tau}&=\frac{\partial f}{\partial\tau}+\{f,B_r\}v_r \\ &=\frac{\partial f}{\partial\tau}-\{f,B_r\}\mathcal{a}_{r4}\frac{\partial\chi_4}{\partial\tau}\,. \end{aligned} \tag{15}$$

We now recall a general result that tells us that this equation of motion can always be put into Hamiltonian form using the DB (13).[16,17] In (15) it is understood that f is some function of Q, P, ξ, η, and τ; it is this last explicit τ dependence that gives the term $\partial f/\partial\tau$ in (15). It is a fact that one can find $6N$ independent variables (in infinitely many ways) which will form a basic set with respect to the DB (13) and with the further property that their DB's with one another are independent of τ *when expressed in terms of themselves again.* Since χ_4 carries explicit τ dependence, these $6N$ quantities might be explicitly dependent on τ as well, when constructed as functions of Q, P, ξ, η. If an f occurring in (15) is now expressed in terms of such a set of $6N$ quantities and τ (and this can be done since *all* the constraints B_r, χ_r are operative), the meaning of the term "explicit dependence on τ" may change.[17] Using the sign $\partial'/\partial\tau$ for the partial derivative in this new sense, we are now assured that a Hamiltonian $\mathcal{H}$ will exist so that in terms of the DB (13) the general equation of motion is

$$\frac{df}{d\tau}=\frac{\partial f}{\partial \tau}-\{f,B_r\}\mathcal{C}_{r4}\frac{\partial \chi_4}{\partial \tau}=\frac{\partial' f}{\partial \tau}+\{f,\mathcal{H}\}^*. \qquad (16)$$

A physically well-motivated choice of the χ_r will be given in Sec. III; subject to that, the physical system of N interacting point particles can at this point be defined. It is a system whose $6N$ independent phase-space variables are the $8N+8$ variables Q, P, ξ_a, η_a modulo the constraints (4), (7), and (10) which can in fact be used as identities. Brackets among general dynamical variables are computed starting from Eq. (13). We have ten generators $J_{\mu\nu}, P_\mu$ to implement changes of inertial frame, and one generator $\mathcal{H}$ to determine dynamical evolution in one frame. The DB's among the former reproduce the Lie algebra of $\mathcal{P}$. The physical interpretation of τ depends on the form of χ_4. In any state of motion, in a given frame θ, a definite set of N world lines can be drawn. All the generators $J_{\mu\nu}, P_\mu$ (and $\mathcal{H}$ too) are constants of motion. From the fact that each V_a is restricted to be a Lorentz-invariant [in the sense of the transformation rules (2)] function of the η_a and the differences of the ξ_a, we easily find

$$\{Q_\mu, D\}=P_\mu/(P^2)^{1/2}, \quad \{Q_\mu, C_\nu\}=0. \qquad (17)$$

It follows that in every state of motion, Q_μ traces a straight world line parallel to P_μ.

III. INVARIANT WORLD LINES—THE WLC

We now take up the question: Are the world lines, whose existence has been secured, invariant? The answer depends on the set of constraints χ_r.

Let θ, θ' be two inertial frames related by an infinitesimal element of $\mathcal{P}$ in this way:

$$x'^\mu=x^\mu+\omega^{\mu\nu}x_\nu+a^\mu, \quad |\omega|, |a|\ll 1. \qquad (18)$$

This is a geometrical connection between the coordinates x, x' assigned in θ, θ' to a single event in space-time. Let the N-particle system be in a state of motion which, in θ, leads to the set of world lines $q_a^\mu(\tau)$: The single point on $\mathcal{C}$ with parameter value τ gives us one point on each particle's world line, and as τ varies along $\mathcal{C}$ these N space-time points trace N paths in space-time. Points in space-time sharing the same τ need not be simultaneous in the physical sense in θ. Now by the definition of the canonical formalism, the space-time coordinates to be used in θ' to build up world lines in that frame, for the same state of motion, are given by

$$q_a'^\mu(\tau)=q_a^\mu(\tau)+\{G, q_a^\mu(\tau)\}^*,$$
$$G=\tfrac{1}{2}\omega^{\mu\nu}J_{\mu\nu}-a^\mu P_\mu. \qquad (19)$$

This is the action of the infinitesimal transformation $R^*(\Lambda, a)$ for the element (18) in $\mathcal{P}$. The set of world lines drawn in θ' will be the same as those which were drawn in θ if each $q_a'(\tau)$ is the geometrical transform, according to (18), of $q_a(\tau+\delta_a\tau)$ for some $\delta_a\tau$. We must only require that the two sets of lines as a whole coincide; it need not happen that N points sharing a common τ value in θ' do so also in θ. The WLC is thus the requirement that there be N infinitesimal expressions $\delta_a\tau$, each linear in $\omega^{\mu\nu}$ and a^μ, such that

$$\{G, q_a^\mu\}^*=\omega^{\mu\nu}q_{a\nu}+a^\mu+\left(\frac{\partial' q_a^\mu}{\partial\tau}+\{q_a^\mu,\mathcal{H}\}^*\right)\delta_a\tau, \qquad (20)$$
$$a=1,2,\ldots,N.$$

This is the condition that the geometrical and canonical rules of transformation for space-time positions be compatible. It is written exclusively in terms of the physical brackets and involves all the eleven generators $J_{\mu\nu}$, P_μ, $\mathcal{H}$.

To check the validity of (20) in any given case, and since $\mathcal{H}$ is difficult to exhibit, we reexpress the WLC in a form in which just the brackets (5) appear:

$$\{q_a^\mu, B_r\}\mathcal{C}_{rs}\{G,\chi_s\}=\{q_a^\mu, B_r\}\mathcal{C}_{r4}\frac{\partial\chi_4}{\partial\tau}\delta_a\tau, \qquad (21)$$
$$a=1,2,\ldots,N.$$

It is clear that suitable expressions $\delta_a\tau$ obeying these WLC can definitely be found, if we choose χ_1, χ_2, χ_3 to be explicitly invariant or form a covariant set under $R(\Lambda, a)$, and χ_4 alone breaks this invariance. For then we have the possibility

$$\delta_a\tau=\{G,\chi_4\}\Big/\frac{\partial\chi_4}{\partial\tau}, \quad a=1,2,\ldots,N \qquad (22)$$

and the invariance of world lines is guaranteed.

One simple choice of χ_r, expressing our original intention that Q should in some sense be the center of the system, is

$$\chi_\mu=\sum_a \epsilon_a\xi_a, \quad \epsilon_a>0, \quad \sum_a\epsilon_a=1,$$
$$\chi_4=P\cdot Q-\tau. \qquad (23)$$

There are just four independent χ's here, and the ϵ_a could be either constants or dynamical variables at this stage. Q then emerges as the center in the sense

$$Q\approx\sum_a\epsilon_a q_a. \qquad (24)$$

We restrict the ϵ_a to be Lorentz scalars, with respect again to $R(\Lambda, a)$. They could depend in any way on the η's, but if we permit them to depend on the ξ's only through the V_a, then because the brackets $\{V_a, C_\mu\}$ vanish we will have the particularly simple relation

$$\{\chi_\mu, C_\nu\} = g_{\mu\nu} - P_\mu P_\nu / P^2 . \tag{25}$$

With the choice (23) for the χ's the WLC (21) are all obeyed if we take

$$\delta_a \tau = -a \cdot P , \quad a = 1, 2, \ldots, N . \tag{26}$$

Note that τ is measured in units of action.

To pin down the ϵ_a further, we examine the limiting case when there are no interactions and all N particles are free. It is reasonable to demand that this situation correspond to $V_a = 0$. In that case, each p_a is a constant of motion and from Eqs. (6), (4), and (23) we find

$$p_a^2 = m_a^2 , \quad P \cdot q_a = \tau , \quad a = 1, 2, \ldots, N . \tag{27}$$

Let us check how the q_a vary with respect to τ. In the equation of motion (15), written as

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \{f, D\} w + \{f, C\} v , \quad P \cdot v = 0 , \tag{28}$$

w and the v's can easily be determined. Since $V_a = 0$, both η_a and ϵ_a are constants of motion; and furthermore,

$$\begin{aligned} &\{\chi_4, D\} = (P^2)^{1/2} , \quad \{\chi_4, C_\nu\} = 0 , \\ &\{\xi_{a\mu}, D\} = \frac{\eta_{a\mu}}{(m_a^2 - \eta_a^2)^{1/2}} , \quad \{\xi_{a\mu}, C_\nu\} = g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} . \end{aligned} \tag{29}$$

From all these relations we find

$$\begin{aligned} &\frac{d\chi_4}{d\tau} = 0 \Rightarrow w = \frac{1}{(P^2)^{1/2}} , \\ &\frac{d\chi_\mu}{d\tau} = 0 \Rightarrow v_\mu = -\frac{1}{(P^2)^{1/2}} \sum_a \frac{\epsilon_a \eta_{a\mu}}{(m_a^2 - \eta_a^2)^{1/2}} . \end{aligned} \tag{30}$$

Therefore, the equation of motion for ξ_a, and hence for q_a, is

$$\begin{aligned} \dot{\xi}_{a\mu} &= \frac{1}{(P^2)^{1/2}} \frac{\eta_{a\mu}}{(m_a^2 - \eta_a^2)^{1/2}} + v_\mu , \\ \dot{q}_{a\mu} &= \frac{1}{(P^2)^{1/2}} \left(\frac{\eta_{a\mu}}{(m_a^2 - \eta_a^2)^{1/2}} + \frac{P_\mu}{(P^2)^{1/2}} \right) + v_\mu \\ &= \frac{p_{a\mu}}{P \cdot p_a} + v_\mu . \end{aligned} \tag{31}$$

For the free case, each q_a must trace a world line parallel to the corresponding (constant) p_a, so we must arrange for v_μ to vanish as a consequence of all the constraints in hand. Given the forms of C_μ in Eq. (7) and v_μ in Eq. (30), we are led to the natural choice

$$\epsilon_a = \frac{(m_a^2 - \eta_a^2)^{1/2}}{\sum_b (m_b^2 - \eta_b^2)^{1/2}} , \quad a = 1, 2, \ldots, N . \tag{32}$$

Q is thus seen as the center of energy[12] (rather than center of mass) of the collection of N free particles. For the interacting case we have two alternatives: (i) We may either retain the choice (32) or (ii) we may make the specific choice

$$\epsilon_a = \frac{(m_a^2 - \eta_a^2 + V_a)^{1/2}}{\sum_b (m_b^2 - \eta_b^2 + V_b)^{1/2}} . \tag{33}$$

More generally, we may choose

$$\begin{aligned} &\epsilon_a = \frac{\phi_a(\eta, m, V)}{\sum_b \phi_b(\eta, m, V)} , \\ &\phi_a(\eta, m, 0) = (m_a^2 - \eta_a^2)^{1/2} . \end{aligned}$$

In either case, the bracket relations (25) are valid. These two possible choices for the ϵ_a must be viewed as leading to two *distinct physical models*.

For this physical system, the following interpretation emerges. In any given state of motion, as seen in the fixed inertial frame θ, the parameter τ is the time in the center-of-momentum frame. That frame is *inertial but dependent on the particular state of motion*. ξ_a is the spatial part of the position of particle a, relative to the center Q, in the center-of-momentum frame; η_a is the spatial part of the momentum of particle a in that frame. The single-particle variables q_a, p_a have no specific significance in the sense of canonical-bracket properties with one another. Their important properties are their Poincaré transformation laws and the numerical values of the q_a in any state of motion. These values lead to invariant world lines. In the free case each p_a is on its mass shell and stays constant, while q_a traces the appropriate straight line uniformly. In any state, even in the presence of interactions, the center Q traces a straight world line

$$Q(\tau) = Q(0) + \tau P/P^2 , \quad P \cdot Q(0) = 0 . \tag{34}$$

IV. DISCUSSION AND OUTLOOK

Dynamical theory of direct particle interactions within a relativistic framework involves a number of demanding conditions. For N particles, interacting or otherwise, we need $6N$ phase-space variables. Relativistic invariance is best incorporated using $2N$ four-vectors for the particles together with $2N$ invariant conditions eliminating $2N$ variables. In this paper we have followed the suggestion of Foldy[4] and Rohrlich[13] to introduce explicitly at the start an $(N+1)$th pair of four-vectors Q, P. The $2N$ second-class transversality conditions (4) immediately reduce the number of independent variables from $8N+8$ to $6N+8$; we then need to get rid of eight more variables. Only then do we arrive at a physical system identifiable with N particles.

This is done in two stages. First, the four constraints (7) allowing us to identify P with the sum of the p_a are adopted. These four are first-class constraints. At this stage with the $2N$ second-class constraints (4) eliminated using the brackets (5), the four first-class constraints define four-dimensional sheets. These sheets are relativistically invariant.[11,13,15] But such a system is *not* identifiable with a collection of particles; instead, we should obtain a one-parameter curve on each sheet describing the dynamical evolution.

To accomplish this we impose four more constraints which together with the four constraints on P make a second-class system of eight constraints, at least one of which is dependent on the evolution parameter τ. *Once these have been imposed we do have a one-parameter curve on each sheet describing the dynamical evolution of a system of N particles.*

The Hamiltonian $\mathcal{H}$ is the generator of the τ evolution and is thus closely related to the fourth first-class constraint and its τ-dependent conjugate. In general it is *not* a generator of the Poincaré group, and in the present theory it is most convenient to choose it not to be a Poincaré generator.

An alternate method of making the reduction from $8N$ phase-space variables to $6N$ phase-space variables in a relativistic invariant way and to introduce interactions has been discussed by us in a recent paper[14] along the lines suggested by Komar.[11] There is no need to introduce the eight new phase-space variables Q, P. Instead, the starting variables are four-vectors q_a, p_a with first-class constraints of the form

$$K_a = P_a{}^2 - m_a{}^2 + V_a(q,p) \approx 0\,, \quad a = 1, 2, \ldots, N\,.$$

If we take a point in the $2N$-dimensional phase space the canonical transformations generated by the N quantities K_a produce an N-dimensional sheet. These sheets are disjoint, and lead to a breakup of the entire phase space into ∞^N such sheets. The theory so developed for two particles is the same as the theory developed by the formalism here. But the two schemes are not fully equivalent when we go beyond two-particle systems. But that formalism is similar to the present theory in having eleven generators.

Canonical formalism was often associated with a Lagrangian theory and a Hamiltonian variational principle. But this is not necessarily so; the canonical formalism and the associated equations of "motion" can stand on their own. The theory of constraint dynamics can also be formulated independent of any Lagrangian.[1] The present study does not involve any Lagrangian; the manifest covariance of the system is implemented in terms of an invariant set of constraints. The world-line conditions are implementable provided the Hamiltonian constraint is relativistically invariant.

The "coordinate" Q describes the uniform motion under the dynamical evolution and may therefore be identifiable with the relativistic center of energy. This happens to be also the proper definition of the center for a collection of free particles to realize the correct relation between velocity and momentum. For interacting particles there is *no unique choice* of how the center is to be constructed; rather, we have a variety of possible choices.

The choice of constraints here is defined invariantly but dynamically; reduction to the unconstrained set of variables is therefore not kinematical but dynamical. This reflects itself in the Poisson-bracket relations (5) and the final brackets (13). Consequently, the framework developed in this paper makes use of a form of relativistic dynamics going beyond Dirac's forms of relativistic dynamics.[1]

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