Gaussian–Maxwell beams

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Gaussian-beam-type solutions to the Maxwell equations are constructed by using results from relativistic front analysis, and the propagation characteristics of these beams are analyzed. The rays of geometrical optics are shown to be the trajectories of energy flow, as given by the Poynting vector. The longitudinal components of the field vectors in the direction of the beam axis, though small, are shown to be essential for a consistent description.

1. INTRODUCTION

In the context of wave optics, scalar Gaussian beams are exact solutions to the paraxial (parabolic) wave equation,[1] and they have received substantial interest because they form the fundamental modes of laser cavities with spherical mirrors. But one knows that light beams consist of electromagnetic-field excitations and hence are governed by Maxwell's equations. In partial recognition of this fact, it is sometimes suggested that the scalar function representing the Gaussian beam should be interpreted as the amplitude of the electric-field vector *assumed to be polarized in the same direction everywhere.*[2] In other words, one imagines a plane-polarized Gaussian beam with the electric vector (and also the magnetic vector) transverse to the beam axis at all points. It is easy to see that such an interpretation is not consistent with Maxwell's equations. The divergence-free nature of the electric field requires that there be no spatial variation of this field vector in the direction of its polarization, but in a Gaussian beam the field amplitude does vary in all directions. Thus it seems desirable to augment the scalar Gaussian beam so that it forms an actual (paraxial) solution to the Maxwell equations.

We presented[3,4] recently in this journal a procedure for describing, in a way consistent with Maxwell's equations, the action of an optical system on the electric- and the magnetic-field vectors, given its action on the field amplitude of conventional scalar optics. This procedure unfolds in a natural way through a relativistic front[5–7] analysis of the Maxwell equations. In the present paper, we use this procedure to derive Gaussian-beam-type solutions to Maxwell's equations. To distinguish these beams from the scalar ones, the former will be called *Gaussian–Maxwell beams.*

We begin in Section 2 by describing the rule for going from scalar to vector optics, derived in Ref. 3. The field in the latter case is to be written as a six-element column vector so that the optical system becomes a 6×6 matrix operator. This is followed in Section 3 by a brief summary of those properties of scalar Gaussian beams that are needed later in this paper. Gaussian–Maxwell beams are constructed and their basic properties analyzed in Section 4. The field vector components parallel to the beam axis, though small, are shown to be essential for a consistent description. Section 5, our final section, contains some concluding discussions.

2. TRANSITION FROM SCALAR TO VECTOR OPTICS

In paraxial scalar optics it is customary to choose the positive z axis as the direction about which the field propagates. The field of a fixed frequency ω is then specified through its (complex) amplitude distribution $U(x_\perp)$ over a transverse plane (z = constant), where $x_\perp = (x, y)$ is a two-dimensional variable and a phase factor $\exp(-i\omega t)$ is suppressed. The effect of an optical system is to map the field distribution in one transverse plane (the input plane) to another plane (the output plane) and is represented by an integral operator $\Omega(x_\perp, P_\perp)$:

$$U_{\text{out}}(x_\perp) = \Omega(x_\perp, P_\perp) U_{\text{in}}(x_\perp),$$
$$P_a = -i\frac{\partial}{\partial x_a}, \qquad a = 1, 2. \tag{2.1}$$

When Ω has no dependence on $P_\perp$, we have the various amplitude transmittances, the thin lens and the Gaussian aperture being special cases. For a thin lens of focal length f we have, for example,

$$\Omega_f(x_\perp) = \exp\left(\frac{-ik}{2f} x_a x_a\right), \tag{2.2}$$

where $k = \omega/c$ and summation over a is implied. On the other hand, for free propagation over a distance d, Ω is a function only of $P_\perp$, and we have[8]

$$\Omega_d(P_\perp) = \exp\left(-\frac{id}{2k} P_a P_a\right). \tag{2.3}$$

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When Eq. (2.2) is used in Eqs. (2.1) and written out in integral form, one obtains the Fresnel propagation formula.[9] When amplitude transmittances are stacked in sequence separated by free-propagation sections, the resulting system has an operator with nontrivial dependence on $x_\perp$ and $P_\perp$. For first-order systems, which incidentally can always be synthesized by using thin lenses and free-propagation sections,[10] this operator reduces to the generalized Huygens integral.[11]

To make the transition to the vector case governed by Maxwell's equations, the field in a transverse plane is written as a six-component column $F(x_\perp)$ consisting of the components of the **E** and **B** vectors in place of the scalar function $U(x_\perp)$:

$$F(x_\perp) = \begin{pmatrix} E_1(x_\perp) \\ E_2(x_\perp) \\ E_3(x_\perp) \\ B_1(x_\perp) \\ B_2(x_\perp) \\ B_3(x_\perp) \end{pmatrix}. \tag{2.4}$$

Then for an optical system whose action on the scalar field is described by $\Omega(x_\perp, P_\perp)$, the corresponding 6×6 matrix operator to act on the vector field $F(x_\perp)$ is obtained by[3] replacing $x_\perp$ in $\Omega(x_\perp, P_\perp)$ by $x_\perp + (1/k)G_\perp$:

$$\Omega(x_\perp, P_\perp) \to \Omega\left(x_\perp + \frac{1}{k} G_\perp, P_\perp\right);$$

$$G_1 = \frac{1}{2}\begin{pmatrix} -S_2 & S_1 \\ -S_1 & -S_2 \end{pmatrix}, \qquad G_2 = \frac{1}{2}\begin{pmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}. \tag{2.5}$$

The important point of course is that $\Omega[x_\perp + (1/k)G_\perp, P_\perp]$ acting on a paraxial $F(x_\perp)$ forming a solution to the Maxwell equations produces another $F(x_\perp)$ that is guaranteed to be a solution to these equations. As noted in Section 1, this rule emerges from a front-form analysis of the Maxwell equations.[12] We have already applied this rule to the action of a thin lens on a paraxial Maxwell field and shown that it gives useful results.[7] Further, we have shown that it leads to an unambiguous generalization of the scalar Fourier optics to the Maxwell-field case.[3] More recently,[4] we have used this rule to describe the action of polarization-sensitive linear optical systems on *arbitrary* paraxial Maxwell fields. In Section 4 of this paper, we adopt this rule to derive Gaussian-beam solutions to the Maxwell equations. But, in Section 3, we collect some results regarding the propagation characteristics of scalar Gaussian beams needed for this purpose.

3. SCALAR GAUSSIAN BEAMS

With the beam propagating along the z axis, we can assume without loss of generality that the waist of the scalar Gaussian beam is in the transverse plane $z = 0$. The field in the waist plane is

$$U(x_\perp; z = 0) = (2/\pi)^{1/2} \frac{1}{\sigma_0} \exp(-x_\perp{}^2/\sigma_0{}^2). \tag{3.1}$$

The field in a later plane z = constant can be computed by using the Fresnel propagation operator [Eq. (2.2)]. One obtains[1,13]

$$\begin{aligned} U(x_\perp; z) &= \exp(-izP_aP_a/2k)U(x_\perp; 0) \\ &= (2/\pi)^{1/2} \frac{1}{\sigma(z)} \exp[-x_\perp{}^2/\sigma(z)^2] \\ &\quad \times \exp[ikx_\perp{}^2/2R(z)], \end{aligned}$$

$$\sigma(z) = \sigma_0\left[1 + \left(\frac{2z}{k\sigma_0{}^2}\right)^2\right]^{1/2},$$

$$R(z) = z\left[1 + \left(\frac{k\sigma_0{}^2}{2z}\right)^2\right]. \tag{3.2}$$

In Eqs. (3.2) we have suppressed an unimportant phase factor independent of the transverse coordinates $x_\perp$. Clearly, $\sigma(z)$ is a measure of the beam width in the general transverse plane and is called the spot size. $R(z)$ is the radius of curvature of the phase front.

In terms of a complex z-dependent parameter q called the complex radius of curvature and defined as[1,13]

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{2i}{k\sigma(z)^2}, \tag{3.3}$$

we can write

$$U(x_\perp; z) = (2/\pi)^{1/2} \frac{1}{\sigma(z)} \exp(ikx_\perp{}^2/2q). \tag{3.4}$$

We will find this form particularly convenient in our analysis in Section 4.

4. GAUSSIAN-MAXWELL BEAMS

To derive Gaussian solutions to the Maxwell equations we first *reinterpret* the scalar field [Eq. (3.1)] as follows. We take a plane wave of unit amplitude propagating exactly along the z axis. Clearly, for such a wave $U(x_\perp) = 1$. At $z = 0$ we pass it through a Gaussian filter with amplitude transmittance

$$\Omega(x_\perp) = (2/\pi)^{1/2} \frac{1}{\sigma_0} \exp(-x_\perp{}^2/\sigma_0{}^2). \tag{4.1}$$

Then what comes out of the filter will be the field [Eq. (3.1)].

With this reinterpretation, the generalization to the vector case is straightforward. Without loss of generality let us take the incident plane wave polarized in the x direction, so that the column $F(x_\perp)$ equals $(1\ 0\ 0\ 0\ 1\ 0)^T$. Using our rule [Eqs. (2.5)], the amplitude transmittance becomes a 6×6 matrix. Thus we have for the field in the plane $z = 0$:

$$F(x_\perp; 0) = (2/\pi)^{1/2} \frac{1}{\sigma_0} \exp\left[-\left(x_\perp + \frac{1}{k} G_\perp\right)^2/\sigma_0{}^2\right] \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.2}$$

On a later transverse plane, the field reads as

$$F(x_\perp; z) = (2/\pi)^{1/2} \frac{1}{\sigma(z)} \exp\left[(ik/2q)\left(x_\perp + \frac{1}{k} G_\perp\right)^2\right] \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.3}$$

The matrix exponential in Eq. (4.3) can be easily evaluated in closed form by virtue of the properties of the matrices G_a. We have[7]

$$G_1 G_2 = G_2 G_1,$$
$$G_a G_a = 0,$$
$$G_a G_b G_c = 0. \tag{4.4}$$

Hence Eq. (4.3) reduces to

$$F(x_\perp; z) = (2/\pi)^{1/2} \frac{1}{\sigma(z)} \exp(ikx_\perp{}^2/2q)$$
$$\times x\left[1 + \frac{i}{q} x_\perp \cdot G_\perp - \frac{1}{q^2}(x_\perp \cdot G_\perp)^2\right] \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.5}$$

Substituting for $G_\perp$ from Eqs. (2.5) we have to the lowest nontrivial order

$$F(x_\perp; z) = (2/\pi)^{1/2} \frac{1}{\sigma(z)} \exp(ikx_\perp{}^2/2q)$$
$$\times \left\{ \mathbb{1} - \frac{1}{2q} \begin{pmatrix} 0 & 0 & -x & 0 & 0 & y \\ 0 & 0 & -y & 0 & 0 & -x \\ x & y & 0 & -y & x & 0 \\ 0 & 0 & -y & 0 & 0 & -x \\ 0 & 0 & x & 0 & 0 & -y \\ y & -x & 0 & x & y & 0 \end{pmatrix} \right\} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.6}$$

In terms of components this reads as

$$E_1(x_\perp; z) = B_2(x_\perp; z) = (2/\pi)^{1/2} \frac{1}{\sigma(z)} \exp(ikx_\perp{}^2/2q),$$
$$E_2(x_\perp; z) = B_1(x_\perp; z) = 0,$$
$$E_3(x_\perp; z) = -\frac{x}{q} E_1(x_\perp; z),$$
$$B_3(x_\perp; z) = -\frac{y}{q} B_2(x_\perp; z). \tag{4.7}$$

These are the **E** and **B** fields of a Gaussian–Maxwell beam. After reinserting the z, t-dependent phase factors, the reader can directly verify that these fields indeed form paraxial solutions to the Maxwell equations.

The transverse components E_1, B_2 propagate in the same way as the scalar beam [Eq. (3.4)], with the waist again at $z = 0$. But the longitudinal components behave differently. Even at $z = 0$, E_3 and B_3 do not have Gaussian dependence on $x_\perp$:

$$E_3(x_\perp; 0) = -(2/\pi)^{1/2} \frac{1}{\sigma_0} \frac{2ix}{k\sigma_0{}^2} \exp(-x_\perp{}^2/\sigma_0{}^2),$$
$$B_3(x_\perp; 0) = -(2/\pi)^{1/2} \frac{1}{\sigma_0} \frac{2iy}{k\sigma_0{}^2} \exp(-x_\perp{}^2/\sigma_0{}^2). \tag{4.8}$$

As they propagate, E_3 and B_3 stay proportional to the respective transverse components and to the transverse distances from the beam axis measured in the direction of polarization of those transverse components. Since q is, in general, complex, the longitudinal components have a part in phase with the transverse ones and a part in phase quadrature. The field is elliptically polarized, **E** in the xz plane and **B** in the yz plane.

Close to the waist plane, i.e., for $z \ll k\sigma_0{}^2/2$, we have

$$\sigma(z) \approx \sigma_0, \qquad R(z) \approx \frac{1}{z}(k\sigma_0{}^2/2)^2,$$

and hence

$$\frac{1}{q} \approx \frac{2i}{k\sigma_0{}^2}.$$

Thus in this region E_3 and B_3 are in phase quadrature with E_1 and B_2, respectively. Therefore **E** and **B** are elliptically polarized with minor axis along the beam axis. Sufficiently far away from the waist plane, i.e., in the Fraunhofer region $z \gg k\sigma_0{}^2/2$, we have

$$q \approx z,$$

so that **E** and **B** are linearly polarized and are transverse to the radial direction.

More insight is gained by computing the Poynting vector. One obtains

$$\mathbf{S} = \mathrm{Re}\, \mathbf{E}^* \wedge \mathbf{B} = \frac{2}{\pi} \frac{1}{\sigma(z)^2} \left[\frac{x_\perp}{R(z)}, 1\right]. \tag{4.9}$$

First, we note that the parts of the longitudinal components that are in phase quadrature with the transverse ones do not contribute to **S**. To see another interesting aspect let us define a two-parameter family of curves

$$x_\perp = (x_\perp)_0 \left[1 + \left(\frac{2z}{k\sigma_0{}^2}\right)^2\right]^{1/2}, \tag{4.10}$$

where the pair of parameters $(x_\perp)_0$ label a curve and z is a parameter along a curve. These curves are well known in the context of scalar Gaussian beams. There they represent the geometrical-optics rays normal to the phase curvature and account for the manner in which the beam diverges. The tangent at any point $(x_\perp, z)$ to the curve passing through that point can be easily calculated by noting from Eqs. (4.10) and (3.2) that

$$\frac{dx_\perp}{dz} = \frac{x_\perp}{R(z)}. \tag{4.11}$$

Thus the tangent vector at $(x_\perp, z)$ is proportional to $[x_\perp/R(z), 1]$, which is exactly the direction of the Poynting vector **S** at that point. We find that the Poynting vector is

along the rays of geometrical optics so that these rays are indeed the trajectories of energy flow. It should be noted that the small longitudinal components are essential for the consistency of the theory because in their absence the energy flow would have been axial everywhere, contrary to the notion of a diverging beam.

To conclude this section we note that it is consistent with Maxwell's equations to have a Gaussian beam with $E_2 = B_1 = 0$. However, it is inconsistent with these equations to require a Gaussian beam for which both E_2 and E_3 vanish and only $E_1 \neq 0$.

5. CONCLUDING DISCUSSION

We have seen that a Gaussian beam polarized in the same direction everywhere is not compatible with Maxwell's equations. The Maxwell field that is closest to the usual scalar Gaussian beam is the Gaussian–Maxwell beam given in Eqs. (4.7). The divergence-free condition on the **E** and **B** vectors imply that *spatial modulation necessarily affects polarization*. In other words, the polarization of the field cannot be chosen independently of its spatial variation. Thus a Gaussian-varying transverse component necessarily results in the longitudinal component in Eqs. (4.7).

Once the choice of polarization of the transverse component is made, the Gaussian–Maxwell beam is completely specified by the complex parameter q, as can be seen from Eqs. (4.7). Thus these beams form a three-parameter family labeled by the orientation of the transverse component of the electric vector (the transverse polarization) and the two real parameters σ and R constituting q or, alternatively, by the transverse polarization, the spot size at the waist, and the location of the waist. Our analysis in Section 4 shows that the transverse polarization does not change under free propagation. It is easy to show that it is also unaffected by action of thin lenses, and, in view of the theorem established in Ref. 7 and already noted in Section 2, it follows that it is an invariant under action by any first-order system. Thus we conclude that the *abcd* law of Kogelnik[14] governing the transformation of q completely describes the transformation properties of all the field components of Gaussian–Maxwell beams under the action of first-order systems.

Gaussian–Schell-model (GSM) beams have received much interest in recent times.[15] It has been shown[16] that GSM beams can be obtained as incoherent superposition of coherent Gaussian beams and their higher-order Hermite–Gaussian beams. In view of this result, extension of our analysis to the GSM case presents no difficulty.

The rule represented in Eqs. (2.5) is evidently central to the later results of the paper. It was established in Ref. 3 through an analysis of the form of the angular-spectrum representation of paraxial vector electromagnetic fields.

Since submission of the manuscript of this paper, the authors' attention has been drawn to possible applications of the analysis to several questions related to problems of wave propagation in inhomogeneous media that are of interest in ocean acoustics, seismology, and other fields. These include construction of the vector wave theories corresponding to the extended parabolic (wide-angle) scalar theories,[17] development of paraxial wave theory for the vector equations of linear elasticity theory,[18] and application to the Gaussian-beam approach for ray-optics computations.[19] We plan to analyze these questions in a subsequent paper.

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