Quantum Measurement and Dynamical Maps

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1 Quantum Measurement and Nonunitary Dynamics

The problem of measurement in a quantum system involves the interaction of a classical system with only a small number of degrees of freedom ('measuring apparatus') coupled to the quantum system which is being subjected to measurement. It has been the practice to think of the measuring apparatus as a quantum system with a very large number of degrees of freedom treated in the classical limit. It is however possible to formulate[1] the problem in such a manner that the measuring appartus is a classical system with a finite number of degrees of freedom.

This formulation involves the perception of the classical system as the projection of a quantum system. Along with a classical system with its phase space ω is a vector field $\partial/\partial\omega$. Together they constitute the phase space Ω of a quantum system.[2] If we proceed to the Schrodinger picture of this phase space and treat ω as the coordinates, the projection is obtained by demanding that only the absolute values of the amplitudes $|\Psi(\omega)|$ are observable. The phases of $\Psi(\omega)$ are unmeasurable. It can now be seen that phase space trajectories in ω are well defined if the hamiltonian in Ω is a vector field in ω. Elsewhere[3] we have analyzed this formulation and shown it to be able to reproduce the traditional Stern-Gelach experiment and other such systems.

It can be seen that the crucial step in passing from superposable complex probability amplitudes to nonnegative standard probabilities involves the loss of phase information at some stage. In the traditional method the phase is lost in the passage to the classical limit of a large quantum system. In our formulation mentioned above it is the postulate of superselection and consequent unobservability of phases[1]. To deal in general with such processes it would be desirable to consider an input-output mechanism where phase information is not necessarily preserved. This is provided by the formalism of dynamical maps which has seen such significant development during the past quarter century.

The method of dynamical maps[4,5] is not a theory but a framework; any successful measurement theory or more generally any non-destructive process which develops a quantum system may be cast as a dynamical map.

2 Dynamical Maps

Let ρ be the density operator of a quantum system, a traceless nonnegative self adjoint operator which specifies the quantum state completely. For any dynamical variable represented by a bounded operator B, the expectation value

$$< B >= \text{tr}\,(\rho B)$$

is assigned. A dynamical map is a linear map from density operator into density operators:

$$\rho \rightarrow \mathcal{L}(\rho): \ \lambda\rho_1 + (1-\lambda)\rho_2 \rightarrow \lambda\mathcal{L}(\rho_1) + (1-\lambda)\mathcal{L}(\rho_2); \ 0 \leq \lambda \leq 1. \tag{1}$$

Clearly such linear maps constitute a convex set: if $\mathcal{L}_1$ and $\mathcal{L}_2$ are dynamical maps, so is

$$\mathcal{L} = \mu\mathcal{L}_1 + (1-\mu)\mathcal{L}_2; \ 0 \leq \mu \leq 1.$$

It would be desirable to find the generating extremal elements of the set of dynamical maps: those for which

$$\mathcal{L} = \mu\mathcal{L}_1 + (1-\mu)\mathcal{L}_2, \ 0 \leq \mu \leq 1 \Longrightarrow \mathcal{L}_1 = \mathcal{L}_2 = L.$$

This is a difficult problem but some progress can be made when ρ is finite dimensional. In the rest of this paper we shall restrict attention to systems whose density operators can be represented by $N \times N$ density matrices.

The conditions on ρ are:

$$\begin{aligned} \rho_{rs} &= \rho^*_{sr} && \text{(hermiticity)}, \\ \text{tr}\,\rho &= 1 && \text{(normalization)}, \\ x^+ \rho x &\geq 0 && \text{(nonnegativity)}. \end{aligned}$$

If the dynamical map is described by

$$\rho_{rs} \to B_{rr',ss'}\rho_{r's'}$$

then

$$\begin{aligned} B_{rr',ss'} &= B_{ss',rr'} \qquad \text{(hermiticity)},\\ B_{rr',rs'} &= \delta_{r's'} \qquad \text{(normalization)}, \\ x_r^* y_{r'}^* B_{rr',ss'} x_s y_{s'} &\leq 0 \text{(nonnegativity)}. \end{aligned} \tag{2}$$

In the pair of indices rr', ss' the $N^2 \times n^2$ matrix B is hermitian and can be written in terms of its eigenvalues:

$$B_{rr',ss'} = \sum_{\alpha=1}^{n^2} \eta(\alpha)\varsigma_{rr'}(\alpha)\varsigma^*_{ss'}(\alpha) \tag{3}$$

but the positivity condition does not guarantee that all the $\eta(\alpha)$ are positive. In the special case $\eta(\alpha) \leq 0$ we refer to the dynamical map as 'completely positive'[6].

Not all dynamical maps are completely positive as seen by the complete classification of all extremal maps by Gorini and Sudarshan[7]. A simple example is given by the map

$$\rho \to \rho^T$$

has a B with eigenvalues $1, 1, 1, -1$. Similarly

$$\rho \to 1 - \rho$$

has a B with eigenvalues $1, 1, 1, -1$.

The completely positive maps form an important subset of maps. In this case by absorbing the numerical value of the nonnegative eigenvalues into the definition of ς we may write:

$$\rho \to \sum_\alpha \varsigma(\alpha)\rho\varsigma^+(\alpha)\ ; \sum_\alpha \varsigma^+(\alpha)\varsigma(\alpha) = 1. \tag{4}$$

It is clear that this map leads to nonnegative normalized density operators. The map is fully characterized by the $\nu \leq N^2$ matrices $\varsigma(\alpha)$. We recognize that if the $\varsigma(\alpha)$ form an acceptable set so do

$$\eta(\alpha) = U(\alpha)\varsigma(\alpha)V^+\ ; \quad \text{no sum on } \alpha. \tag{5}$$

Hence V and $U(\alpha)$ are $\nu+1$ unitary matrices with are otherwise unrestricted.

Consider the Kronecker product $\rho \times \chi$ where $\chi_{\alpha\beta}$ is $\nu \times \nu$. Let $/W_{r\alpha,s\beta}$ be any $N\nu \times N\nu$ unitary matrix in the Kronecker product space. Then the map

$$\rho \times \chi = R \rightarrow WRW^{+} \tag{6}$$

is unitary. In explicit index notation

$$R_{r\alpha,s\beta} \rightarrow W_{r\alpha,\alpha' r} R_{r' r, s'\delta} (W_{s}\beta, s'\delta)^{*}.$$

we now take the partial trace

$$\rho'_{rs} = R_{r\alpha,s\alpha}.$$

Then

$$\rho_{rs} \rightarrow \rho'_{rs} = W_{r\alpha,r'\beta} (W_{s\alpha,s'\delta})^{*} \chi_{\beta\delta} \rho_{r's'}$$

is a dynamical map. In particular if only $\chi_{1,1}$ is unity with all other matric elements vanishing

$$\rho_{rs} \rightarrow \rho'_{r,s} = W_{r\alpha,r'1} (W_{s\alpha,s'1})^{*} \rho_{r's'}. \tag{7}$$

This is of the form of a completely positive map with

$$\varsigma_{rr'}(\alpha) = W_{r\alpha,r'1}. \tag{8}$$

Conversely, ***given any completely positive map we can consider it as the restriction of a unitary evolution for an enlarged system.*** Clearly the enlarged system need have only $n\nu$ states: the 'reservoir' need have only dimension ν. The dynamics of the enlarged system is not unique since only the subset $W_{r\alpha,r'1}$ enter the characterization of the dynamical map.

3 Characterization of Extremal Maps

We now proceed to characterize the extremal dynamical maps. Towards this end we recognize that the matrices ς form an inner product vector space ν with scalar product.

$$(\varsigma(\alpha), \varsigma(\beta)) = \varsigma^{*}_{rs}(\alpha)\varsigma_{rs}(\beta) \tag{9}$$

with dimension N^2. By choice of ς, distinct values α, β make them orthogonal. The map (4) is invariant under the orthogonal group

$$\varsigma(\alpha) \to \sum_\beta M_{\alpha\beta}\varsigma(\beta) \ , \tag{10}$$

where M is any orthogonal matrix. We shall use (5) and (10) to simplify the extremal elements.

We note, first of all that if and only if the map (4) is not extremal then the ν^2 matrices $\varsigma(\alpha)\varsigma^+(\beta)$ are not linearly independent[6]. The 'only if' part is easy to prove: if

$$m_{\alpha\beta}\varsigma^+(\alpha)\varsigma(\beta) = \mu_\alpha\varsigma^+(\alpha)\varsigma(\alpha) = 0 \ , \tag{11}$$

(without loss of generality m assumed to be hermitian and diagonal !) then

$$\frac{1}{2}(1+\mu_\alpha)\varsigma(\alpha)\rho\varsigma^+(\alpha) + \frac{1}{2}(1-\mu_\alpha)\varsigma(\alpha)^+(\alpha) = \varsigma(\alpha)\rho\varsigma^+(\alpha).$$

But

$$\varsigma^I(\alpha) = \sqrt{1+\mu_\alpha}\varsigma(\alpha) \quad , \varsigma^{II}(\alpha) = \sqrt{1-\mu_\alpha}\varsigma(\alpha)$$

both satisfy (4) and they define dynamical maps. To prove the 'if' part, we recognize that if

$$\lambda\varsigma^I(\alpha)\rho\varsigma^{I+}(\alpha) + (1-\lambda)\varsigma^{II}(\alpha)\rho\varsigma^{II+}(\alpha)$$
$$= \varsigma(\alpha)\rho\varsigma^+(\alpha)$$

then the matrices ς span both ς^I, ς^{II}. It is therefore possible to write

$$\varsigma^I(\alpha) = a_{\alpha\beta}\varsigma(\beta) \ ,$$
$$\varsigma^{II}(\alpha) = b_{\alpha\beta}\varsigma(\beta) \ ;$$

and, by virtue of (4),

$$a^*_{\gamma\alpha}a_{\gamma\beta}\varsigma^+(\alpha)\varsigma(\beta) = 1,$$
$$b^*_{\gamma\alpha}b_{\gamma\beta}\gamma^+(\alpha)\varsigma(\beta) = 1.$$

Choosing

$$m_{\alpha\beta} = a^*_{\gamma\alpha}a_{\gamma\beta} - b^*_{\gamma\alpha}b_{\gamma\beta}$$

we have proved sufficiency.

Since no more than N matrices can fail to satisfy (11) it follows that for every extremal map $\nu \leq N$.

4 The General Form of Extremal Completely Positive Maps

The case $\nu = 1$ is the standard unitary map since (4) reduces to the unitarity condition. So the first nontrivial extremal completely positive dynamical map is for $\nu = 2$.

$$\varsigma^{+}(1)\varsigma(1) + \varsigma^{+}(2)\varsigma(2) = 1.$$

Choose $U(2) = V$ to diagonalize $\varsigma(2)$:

$$\varsigma(2) = \begin{pmatrix} \cos\theta_1 & 0 \\ 0 & \cos\theta_2 \end{pmatrix}.$$

Then

$$\varsigma^{+}(1)\varsigma(1) = \begin{pmatrix} \sin^2\theta_1 & 0 \\ 0 & \sin^2\theta_2 \end{pmatrix}.$$

By choosing $U(1)$ properly we could transform $\varsigma(1)$ to obtain

$$\varsigma(1) = \begin{pmatrix} \sin\theta_1 & 0 \\ 0 & \sin\theta_2 \end{pmatrix}$$

Therefore the most general extremal completely positive dynamical map is of the form

$$\varsigma^{0}(1) = \begin{pmatrix} \sin\theta_1 & 0 \\ 0 & \sin\theta_2 \end{pmatrix}$$

$$\varsigma^{0}(2) = \begin{pmatrix} \cos\theta_1 & 0 \\ 0 & \cos\theta_2 \end{pmatrix} \tag{12}$$

together with transformation of the type (5).

For larger values of ν we may proceed by a modification of this method. We could use $U(\nu) = V$ to diagonalize $\varsigma(\nu)$ to obtain

$$\varsigma(\nu) = \begin{pmatrix} \cos\theta_1 & 0 & \cdot & \cdot & \cdot \\ 0 & \cos\theta_2 & \cdot & \cdot\cdot & \\ & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cos\theta_\nu \end{pmatrix}$$

It follows that

$$\varsigma^{+}(1)\varsigma(1)+\varsigma^{+}(2)\varsigma(2)+\ldots+\varsigma^{+}(\nu-1)\varsigma(\nu-1)=\begin{pmatrix}\sin^2\theta_1 & 0 & \ldots\\ 0 & \sin^2\theta_2 & \ldots\\ \ldots & \ldots & \sin^2\theta_2\end{pmatrix}.$$

So if we define

$$\varsigma(\mu)=\varsigma'(\mu)\begin{pmatrix}\sin\theta_1 & 0\vdots & & \\ 0 & \sin\theta_2 & \ddots & \\ \cdot & \ldots & \ldots & \sin\theta_2\end{pmatrix},$$

it follows that

$$\varsigma'(1)\varsigma'(1)+\varsigma^{+\prime}(2)\varsigma'+\ldots+\varsigma'^{+}(\nu-1)\varsigma'(\nu-1)=1.$$

[If some $\sin\theta$ vanishes we may define the k-the column of all $\varsigma(\mu)$ arbitarily without changing the results !]. The problem is reduced to the question of constructing $\nu-1$ matrices $\varsigma'(\alpha)$.

5 Discussion

In this paper we have outlined the problem of dynamical maps with special attention to completely positive maps and their extremals. In all cases these maps could be considered as contractions of the unitary development of a larger system. The larger system may be thought of as the system under study being coupled to a reservior. The reservior need not be very large: in fact in the general case it need be only $\nu\times\nu\le N^2\times N^2$ and in the case of extremal maps only $\nu\times\nu\le N\times N$. The general form of $\varsigma(\alpha)$ in the extremal case is:

$$\begin{aligned}\varsigma(\nu)&=U(\nu;\nu)C_\nu V(\nu)\ ;\\ \varsigma(\nu-1)&=U(\nu-1;\nu)S_\nu U(\nu-1;\nu-1)C_{\nu-1}V(\nu-1)\ ;\\ \varsigma(\nu-2)&=U(\nu-2;\nu)S_\nu U(\nu-2;\nu-1)S_{\nu-1}\\ &\quad U(\nu-2;\nu-2)C_{\nu-2}V(\nu-2)\ ;\\ \varsigma(\nu-3)&=U(\nu-3;\nu)S_\nu U(\nu-2;\nu-1)S_{\nu-1}\\ &\quad U(\nu-2;\nu-2)S_{\nu-2}U(\nu-3;\nu-3)C_{\nu-3}V(\nu-3)\end{aligned}\tag{14}$$

and so on. Here C_μ, S_μ are the diagonal matrices.

$$C_\mu = \begin{pmatrix} \cos\theta_{1,\mu} & & \\ & \ddots & \\ & & \cos\theta_{\nu,\mu} \end{pmatrix}$$

$$S_\mu = \begin{pmatrix} \sin\theta_{1,\mu} & & \\ & \ddots & \\ & & \sin\theta_{\nu,\mu} \end{pmatrix} \tag{15}$$

$U(\alpha;\beta), V(\beta)$ are suitable unitary matrices.

Any scheme for measurement and loss of phase information or partial loss of phase information must fall under this classification.

Other aspects like viewing quantum measurement as a problem in symmetry breaking have interested Yu'val Ne'eman; it is in honor of his sixtieth birthday that this article is contributed.

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