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Inequivalent Quantizations in Multiply Connected Spaces (*) (**).

P. A. HORVATHY
Department of Mathematics, University of Metz - Metz, France

G. MORANDI
Dipartimento di Fisica dell'Università - Ferrara, Italia
GNSM-CISM e INFN - Sezione di Ferrara, Italia

E. C. G. SUDARSHAN
Center for Particle Theory, University of Texas at Austin - Austin, Texas
The Institute of Mathematical Sciences - Madras, India

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Summary. — The novel features which show up in the process of quantization of a dynamical system on a multiply (nonsimply)-connected configuration space are analysed in the present paper. After rediscussing the path integral approach to the problem, we show how one can give a fiber bundle classification of the inequivalent quantizations associated with nonsimply connected spaces. We discuss various examples, and the generalization to the case in which, due to internal symmetries, a system can admit of nonscalar quantizations.

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1. – Introduction.

The path integral formulation of quantum mechanics [1,2] is the one in which one can see in the most natural and transparent way that the quantization of a

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(**) To speed up publication, the authors have agreed not to receive proofs which have been supervised by the Scientific Committee.
(1) R. P. FEYNMAN and A. R. HIBBS: *Quantum Mechanics and Path Integrals* (McGraw Hill, New York, N.Y., 1965).
(2) L. S. SCHULMAN: *Techniques and Applications of Path-Integration* (J. Wiley, New York, N.Y., 1981).

given classical dynamical system depends often in a crucial way upon the global topological properties of the underlying configuration space. Indeed, the propagator is expressed, roughly speaking, as a weighed sum over all possible paths connecting any two given points in configuration space. As the paths wander through all of the space, they also probe its topology.

To be specific, let $\boldsymbol{Q}$, $\dim \boldsymbol{Q} = n$, be the configuration space of a given classical dynamical system. We shall assume, for the time being, that the system can be quantized according to «scalar» quantum mechanics, *i.e.* that its fixed-time quantum states are given by square-integrable, complex-valued wave functions:

$$(1.1) \qquad \psi(q) \equiv \langle q|\psi\rangle\,; \qquad q \in \boldsymbol{Q}\,; \qquad \psi \in L_2(\boldsymbol{Q}, \boldsymbol{C}, \mathrm{d}\mu)\,,$$

with respect to a given measure $\mathrm{d}\mu$ on $\boldsymbol{Q}$. The most common case is $\boldsymbol{Q} = \boldsymbol{R}^n$ and $\mathrm{d}\mu$ the Lebesgue measure. If $|q\rangle$ and $|q'\rangle$ are two eigenstates of the position operator, the transition amplitude $\langle q, t|q', t'\rangle$ that the state of the system be $|q\rangle$ at time t if it was $|q'\rangle$ at time t' is expressed in path integral form (1,2) as

$$(1.2) \qquad \langle q, t|q', t'\rangle := K(q, t; q', t') = \int \mathscr{D}[\gamma] \exp\left[\frac{i}{\hbar} S[\gamma]\right], \qquad t > t'\,.$$

Here, $\mathscr{D}[\gamma]$ stands for the integration «measure» on the space of all paths joining q' to q:

$$(1.3) \qquad \gamma\colon [t', t] \to \boldsymbol{Q}\,, \qquad \gamma(t') = q'\,, \qquad \gamma(t) = q$$

and the «Feynman factor» $\exp[(i/\hbar)\, S[\gamma]]$ weighing the path γ is defined in terms of the classical action evaluated along the path, *i.e.*

$$(1.4) \qquad S[\gamma] := \int\limits_{t'}^{t} \mathrm{d}\tau\, \mathscr{L}(q(\tau), \dot{q}(\tau), \tau)\,,$$

where $\mathscr{L}(q, \dot{q}, \tau)$ is the classical Lagrangian. In more intrinsic terms, one can introduce, on $T\boldsymbol{Q} \times \boldsymbol{R}$, with $T\boldsymbol{Q}$ the tangent bundle over $\boldsymbol{Q}$ (3,4), the one-form

$$(1.5) \qquad \theta_{\mathscr{L}} := p_i \mathrm{d}q^i - H\, \mathrm{d}t\,,$$

where

$$(1.6) \qquad p_i := \frac{\partial \mathscr{L}}{\partial \dot{q}^i}\,, \qquad H := p_i \dot{q}^i - \mathscr{L}\,.$$

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(3) R. ABRAHAM and J. R. MARSDEN: *Foundations of Mechanics* (Benjamin, 1978).
(4) G. MARMO, E. J. SALETAN, A. SIMONI and B. VITALE: *Dynamical Systems. A Geometric Approach to Symmetry and Reduction* (J. Wiley, New York, N.Y., 1985).

Defining the lift $\tilde{\gamma}$ of the path γ to $T\boldsymbol{Q}\times\boldsymbol{R}$ as

$$\tilde{\gamma}:[t',t]\rightarrow T\boldsymbol{Q}\times\boldsymbol{R} \qquad \text{by} \quad \tau\rightarrow(q(\tau),\dot{q}(\tau),\tau)\,, \tag{1.7}$$

one can easily convince oneself that

$$S[\gamma]=\int_{\tilde{\gamma}}\theta_{\mathscr{L}}\,. \tag{1.8}$$

Remark. If $\mathscr{L}$ is a regular Lagrangian, *i.e.* if the Legendre map [3] from $T\boldsymbol{Q}$ to the cotangent bundle $T^*\boldsymbol{Q}$

$$\boldsymbol{F}\mathscr{L}:T\boldsymbol{Q}\rightarrow T^*\boldsymbol{Q} \qquad \text{by} \quad (q^i,\dot{q}^i)\rightarrow(q^i,p_i) \tag{1.9}$$

is a (local) diffeomorphism, then the r.h.s. of (1.5) can be read off as a one-form Θ_H on $T^*\boldsymbol{Q}\times\boldsymbol{R}$, with H the Hamiltonian of the system. Θ_H is the Cartan form [5,6] associated with H, and $\theta_{\mathscr{L}}$ is related to it via the pull-back:

$$\theta_{\mathscr{L}}=(\boldsymbol{F}\mathscr{L})^*\,\Theta_H\,. \tag{1.10}$$

As is well known, the correspondence between quantum states and wave functions is not one to one. As all physical predictions depend upon quantities like $\psi^*\psi$, a (pure) quantum state is actually defined up to a phase which may be a function on $\boldsymbol{Q}$, *i.e.* $\psi(q)$ and $\exp[(i/\hbar)\,\alpha(q)]\cdot\psi(q)$, with $\alpha\in\mathscr{F}(\boldsymbol{Q})$ (the set of smooth functions on $\boldsymbol{Q}$) having the dimension of an action, define the same quantum state. Multiplying the states by phase factors of this sort leads to

$$K(q,t;q',t')\rightarrow\exp\left[\frac{i}{\hbar}\phi(q,q')\right]\cdot K(q,t;q',t')\,, \tag{1.11}$$

where

$$\phi(q,q'):=\alpha(q')-\alpha(q)\,. \tag{1.12}$$

Vice versa, looking directly at the path integral expression for the propagator $K(q,t;q',t')$, one might think, on the same grounds, that K is ambiguous by a phase factor of the form (1.11), but with ϕ any function of its arguments. However, the semigroup property [1,2]

$$K(q,t;q',t')=\int\mathrm{d}\mu(q'')\,K(q,t;q'',t'')\cdot K(q'',t'';q',t')\,, \qquad t>t''>t'\,, \tag{1.13}$$

(5) E. Cartan: *Leçons sur les Invariants Integraux* (Hermann, Paris, 1922).
(6) J. M. Souriau: *Structure des Systèmes Dynamiques* (Dunod, Paris, 1969).

will be preserved iff ϕ satisfies

$$\phi(q, q'') + \phi(q'', q') = \phi(q, q') \bmod 2\pi\hbar\,, \qquad \forall q, q', q''\,, \tag{1.14}$$

and this will force ϕ to be of the form given in eq. (1.12) for some $\alpha \in \mathscr{F}(\boldsymbol{Q})$.

The above phase ambiguities in the wave function and/or the propagator correspond to the so-called «gauge freedom» in the Lagrangian, *i.e.* to the possibility of adding to $\mathscr{L}$ a total time derivative without altering the classical dynamics [7] or, equivalently, to that of altering $\theta_{\mathscr{L}}$ by the addition of a closed (actually exact) one-form:

$$\mathscr{L} \sim \mathscr{L} - \dot{q}^i \frac{\partial \alpha}{\partial q^i}, \qquad \theta_{\mathscr{L}} \sim \theta_{\mathscr{L}} - \mathrm{d}\alpha\,, \tag{1.15}$$

where the « $\sim$ » sign means that both Lagrangians (or Lagrangian one-forms) yield, via the action principle [7], the same classical equations of motion.

Remark. In elementary quantum mechanics it is possible to accommodate the phase ambiguities in the definition of a quantum state by considering wave functions as functions on an enlarged space $\hat{\boldsymbol{Q}}$, which looks locally as $\boldsymbol{Q} \times \boldsymbol{U}(1)$. If $\hat{q} \in \hat{\boldsymbol{Q}}$ identified by local coordinates $(q, \exp[i\,\alpha])$, there is a natural action of $\boldsymbol{U}(1)$ on $\hat{\boldsymbol{Q}}$ defined by

$$\hat{q} \to \hat{q}\exp[i\theta] := (q, \exp[i(\alpha + \theta)]\,, \qquad \exp[i\theta] \in \boldsymbol{U}(1)\,. \tag{1.16}$$

(Note that, for simplicity, we are calling here α, what was previously denoted by $\alpha/\hbar$.) In order to preserve the fact that $\psi^*\psi$ should be a well-defined function on $\boldsymbol{Q}$ (stated otherwise, $\psi^*\psi$ should be *projectable* onto $\boldsymbol{Q}$), the wave functions: $\psi = \psi(\hat{q})$ are requested to satisfy:

$$\psi(\hat{q}\exp[i\theta]) = \exp[i\theta] \cdot \psi(\hat{q}) \tag{1.17}$$

(less restrictive conditions can be devised, but they will not alter the essence of the structure we are sketchily describing here). A more thorough discussion [8-10] shows that the resulting structure of $\hat{\boldsymbol{Q}}$ is that of a principal $\boldsymbol{U}(1)$ bundle over

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(7) E. C. G. SUDARSHAN and N. MUKUNDA: *Classical Dynamics. A Modern Perspective* (J. Wiley, New York, N.Y., 1974).

(8) F. ZACCARIA, E. C. G. SUDARSHAN, J. S. NILSSON, N. MUKUNDA, G. MARMO and A. P. BALACHANDRAN: *Phys. Rev. D*, **27**, 2327 (1983).

(9) A. P. BALACHANDRAN: in *Geometric and Algebraic Aspects of Nonlinear Field Theories, Amalfi, 1988*, to appear.

(10) E. C. G. SUDARSHAN: in ref. (9).

$\boldsymbol{Q}$ [11] and that wave functions are sections [11] of the bundle satisfying the equivariance property (1.17). The bundle may be or may not be trivial. In the former case (and only in that case), *i.e.* when $\boldsymbol{Q}$ is globally diffeomorphic to the product bundle $\boldsymbol{Q} \times \boldsymbol{U}(1)$, there exist [11] global sections. Hence, a global choice of phases is possible and one recovers in this way the conventional description in terms of wave functions on $\boldsymbol{Q}$. In general, however, the bundle will turn out to be nontrivial, *i.e.* «twisted» in some sense. No global sections will then exist and the quantum-mechanical description will have to be done forcedly in the language of fiber bundles and sections thereof. A relevant example of this sort is provided by the (quantum) dynamics of a charged particle in the field of a Dirac monopole [8].

The same kind of construction, namely the «unfolding» of $\boldsymbol{Q}$ into a $\boldsymbol{U}(1)$ bundle occurs also in the discussion of certain *classical* dynamical systems (the charge-monopole system being again one of them) when one looks for a space on which a global Lagrangian description can be achieved [8,12,13]. We also recall that complex line bundles (*i.e.* vector bundless associated [11] with principal $\boldsymbol{U}(1)$ bundles) with connection play a central role in the development of the geometric quantization programme [6,14,15].

If the configuration space $\boldsymbol{Q}$, which will always be assumed to be path-connected and locally path connected [16], is also simply connected, *i.e.* if the first homotopy group [16] $\pi_1(\boldsymbol{Q})$ is trivial, the global phase ambiguities discussed above are the only ones which are allowed in the quantum-mechanical description [17,18]. As has been recently discussed [19], the same will hold true, *i.e.* there will be no essential quantization ambiguities, if $\pi_1(\boldsymbol{Q})$ is a «perfect» group [20], that is if π_1 is isomorphic to its own commutator subgroup:

$$\pi_1 \sim [\pi_1, \pi_1]\,, \tag{1.18}$$

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(11) Y. Choquet-Bruhat, C. Morette DeWitt (with M. Dillard-Bleick): *Analysis, Manifolds and Physics* (North-Holland, Amsterdam, 1982).
(12) A. P. Balachandran, G. Marmo, B. S. Skagerstam and A. Stern: *Gauge Symmetries and Fiber Bundles* in *Lecture Notes in Physics*, Vol. **188** (Springer, Berlin, 1983).
(13) G. Marmo, G. Morandi and C. Rubano: in *Symmetries in Physics*, edited by B. Gruber, to appear.
(14) B. Konstant: in *Lecture Notes in Mathematics*, Vol. **170** (Springer, Berlin, 1970).
(15) D. J. Simms and N. Mj. Woodhouse: *Lectures on Geometric Quantization* in *Lecture Notes in Physics*, Vol. **53** (Springer, Berlin, 1976).
(16) F. H. Croom: *Basic Concepts of Algebraic Topology* (Springer, Berlin, 1978).
(17) W. Pauli: *Helv. Phys. Acta*, **12**, 247 (1939) and *Enc. Phys.*, **5**, 45 (1958).
(18) E. Merzbacher: *Am. J. Phys.*, **30**, 237 (1962).
(19) T. D. Imbo and E. C. G. Sudarshan: *Phys. Rev. Lett.*, **60**, 481 (1988).
(20) D. J. S. Robinson: *A Course in the Theory of Groups* (Springer, Berlin, 1982).

where $[\pi_1, \pi_1]$, the commutator subgroup, is the minimal subgroup of π_1 containing all the elements of the form $g \cdot h \cdot g^{-1} \cdot h^{-1}$, for $g, h \in \pi_1$.

In general, however, if π_1 is not perfect (in particular, not trivial), there will arise further quantization ambiguities, which will give rise to inequivalent (*i.e.* not unitarily equivalent) quantization schemes for the same underlying classical system. Schulman ([21]), Laidlaw and Morette DeWitt ([22]), and Dowker ([23]) have proved a general theorem for the form of the Feynman propagator when π_1 is not trivial, which we will state here, and rederive in the next section.

Picking up a fiducial point $q_0 \in \boldsymbol{Q}$, one can build up a «homotopy mesh» ([22]) based at q_0 by connecting q_0 to each point $q \in \boldsymbol{Q}$ via a fiducial path $\gamma_0(q)$. In this way, any path γ connecting any two points $q', q \in \boldsymbol{Q}$ can be made to correspond to a loop $L(\gamma)$ based at q_0, defined by

$$L(\gamma) := \gamma_0(q') \cdot \gamma \cdot \gamma_0(q)^{-1} \tag{1.19}$$

and hence, though in a noncanonical way, to an element of the fundamental (first homotopy) group π_1. In ref. ([21-23]) it was then proved that the most general form of the Feynman propagator is, «modulo» an overall phase factor,

$$K(q, t; q', t') = \sum_{g \in \pi_1} \chi(g) \cdot K_g(q, t; q', t'), \tag{1.20}$$

where K_g is the partial propagator obtained by restricting the path integral to the homotopy class of paths labelled by the element $g \in \pi_1(\boldsymbol{Q})$ and $\chi(g)$ is a one-dimensional unitary representation, or a *character*, of the fundamental group

$$\chi^*(g) \cdot \chi(g) \equiv 1, \qquad \chi(g) \cdot \chi(g') = \chi(g \cdot g'), \qquad \forall g, g' \in \pi_1(\boldsymbol{Q}). \tag{1.21}$$

Changing the homotopy mesh changes of course the labelling of paths. As discussed in ref. ([22]), however, this only alters the r.h.s. of (1.20) by an overall phase factor, a harmless ambiguity which we shall agree to ignore henceforth.

Theorem (1.20) was proved by Schulman for the case $\pi_1 = \boldsymbol{Z}$ and $\pi_1 = \boldsymbol{Z}_2$, by Laidlaw and Morette-de Witt for a general, but finite, fundamental group, and by Dowker in the general case. Its main content is that there will be as many different (scalar) quantizations for a given classical system as there are distinct characters of the fundamental group.

That the motivations for studying quantum mechanics in multiply connected

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([21]) L. S. SCHULMAN: *J. Math. Phys. (N.Y.)*, **12**, 304 (1971); see also ref. ([2]).
([22]) M. G. G. LAIDLAW and C. MORETTE DEWITT: *Phys. Rev. D*, **3**, 1375 (1971).
([23]) J. S. DOWKER: *a) J. Phys. A*, **5**, 936 (1972); *b) Selected Topics in Topology and Quantum Field Theory* (Austin Lectures, 1979), unpublished; *c) J. Phys. A*, **18**, 3521 (1985).

(or nonsimply connected) spaces are of more than a passing academic interest can be seen from the following, by no means exhaustive, list of examples:

i) In an idealized set-up of the Aharonov-Bohm effect ([24-29]), the configuration space for a charged particle moving in the presence of a long, impenetrable solenoid may be taken to be $\boldsymbol{Q}=\boldsymbol{R}^2-\Delta$, with Δ a closed disk in the plane enclosing the solenoid's magnetic flux, which is topologically equivalent to the punctured plane $\boldsymbol{R}^2-(0)\simeq \boldsymbol{S}^1\times\boldsymbol{R}$. In this case $\pi_1(\boldsymbol{Q})=\boldsymbol{Z}$, and the nontrivial topology can also be viewed as arising from the boundary conditions imposed on wave functions, namely vanishing boundary conditions on $\partial\Delta$, the boundary of Δ.

ii) n identical particles moving in $\boldsymbol{R}^d$ can be described ([22,30]) in the configuration space:

$$\boldsymbol{Q}=[\boldsymbol{R}^{nd}-\boldsymbol{D}]/\boldsymbol{S}^n \tag{1.22}$$

where $\boldsymbol{D}$ is the diagonal

$$\boldsymbol{D}=((\boldsymbol{x}^1,\dots\boldsymbol{x}^n)\in\boldsymbol{R}^{nd}\ \text{such that}\ \boldsymbol{x}^i=\boldsymbol{x}^j\ \text{for at least a pair}\ i\neq j) \tag{1.23}$$

and $\boldsymbol{S}^n$ is the permutation group on n elements. $\boldsymbol{Q}$ is nonsimply connected, with $\pi_1(\boldsymbol{Q})=\boldsymbol{S}^n$ for $d\geqslant 3$, and with a more complicated fundamental group (the $(n-1)$-string braid group ([31,32])) for $d=2$.

iii) A rigid body with a fixed point has $\boldsymbol{Q}=\boldsymbol{SO}(3)$ ([7]). If the body is free, then $\boldsymbol{Q}=\boldsymbol{SO}(3)\times\boldsymbol{R}^3$, but the addition is topologically trivial. In any event, $\pi_1(\boldsymbol{Q})=\boldsymbol{Z}_2$, leading to the possibility of quantizing the motion either as that of a boson or as that a fermion ([33]).

Finally, let us recall that many interesting aspects of field theory (both at the

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([24]) Y. Aharonov and D. Bohm: *Phys. Rev.*, **115**, 485 (1959).
([25]) T. T. Wu and C. N. Yang: *Phys. Rev. D*, **12**, 3845 (1975).
([26]) M. V. Berry: *Eur. J. Phys.*, **1**, 240 (1980).
([27]) P. A. Horvathy: *a*) *Phys. Lett. A*, **76**, 11 (1980); *b*) in *Differential Geometric Methods in Math. Phys.*, *Lecture Notes in Mathematics*, Vol. **836** (Springer, Berlin, 1979); *c*) *Lecture Notes in Mathematics*, Vol. **905** (Springer, Berlin, 1982); *d*) *Phys. Rev. A*, **31**, 1151 (1985); *e*) *Europhys. Lett.*, **2**, 195 (1986); *f*) *Quantum Ambiguities*, Preprint (Metz, 1988).
([28]) G. Morandi and E. Menossi: *Eur. J. Phys.*, **5**, 49 (1984).
([29]) S. Olariu and I. I. Popescu: *Rev. Mod. Phys.*, **57**, 339 (1985).
([30]) F. J. Bloore: in ref. ([27b]).
([31]) Y. S. Wu: *Phys. Rev. Lett.*, **52**, 2103 (1984).
([32]) *a*) E. C. G. Sudarshan, T. D. Imbo and T. R. Govindarayan: *Center for Particle Theory* (Austin, Preprint DOE-ER40200-137, April 1988); *b*) E. C. G. Sudarshan, T. D. Imbo and C. S. Imbo: *Center for Particle Theory* (Austin, Preprint DOE-ER40200-140, May 1988).
([33]) L. S. Schulman: *Phys. Rev.*, **176**, 1558 (1968).

classical and at the quantum level) like, *e.g.*, θ-vacua ([34]) and Yang-Mills instantons ([11,34,35]) also arise from the nontrivial topology of the underlying configuration space. For a comprehensive review of such effects in quantum mechanics and field theory, see ref. ([23b,36]). In condensed-matter physics, nontrivial topological effects have been found and extensively studied in the classification of defects in ordered media ([37]). θ-vacua–type structures appear in the description of the electrodynamics of superconducting junctions ([38]), and, finally, two-dimensional models of strongly-correlated electrons in high magnetic fields describing the fractionally-quantized Hall effect have been found ([39,40]) to obey fractional statistics, the latter being again of topological origin.

The paper, which is partly review in character, is organized as follows.

In sect. **2** we review the general formulation of quantum mechanics in nonsimply connected spaces, relating the formulation to the quantum-mechanical description on the (simply connected) universal covering space. In this context, we give a brief rederivation of the basic theorem of eq. (1.20). The results of sect. **2** are recasted in the language of fiber bundles and connections in sect. **3**, where we also discuss the relationship between the classification scheme described in sect. **2** and that of «prequantum bundles» wich occurs in geometric quantization. In sect. **4** we discuss some examples and, in the final section, generalizations to nonscalar quantum mechanics. In this last section, we also discuss the results obtained previously, and draw some conclusions.

2. – Quantum mechanics in nonsimply-connected spaces.

Let $\boldsymbol{Q}$ be the configuration space of a classical system, which we will assume to be quantizable according to scalar quantum mechanics. $\boldsymbol{Q}$ is assumed to be path connected and locally path connected, but nonsimply connected, *i.e.* $\pi_1(\boldsymbol{Q}) \neq (e)$. The probabilistic interpretation of quantum mechanics will be preserved if taking a wave function $\psi(q)$ around a generic loop γ based at q alters it at most by a phase factor, *i.e.* if

$$(\hat{\gamma}\cdot\psi)(q) = a(\gamma)\cdot\psi(q)\,, \qquad a(\gamma)\,a(\gamma)^* = 1\,, \tag{2.1}$$

([34]) R. Jackiw: in *Relativity, Groups and Topology*, edited by B. S. De Witt and R. Stora (Elsevier, Houston, Tex., 1984).
([35]) R. Rajaraman: *Solitons and Instantons* (North-Holland, Amsterdam, 1982).
([36]) C. J. Isham: in ref. ([34]).
([37]) N. D. Mermin: *Rev. Mod. Phys.*, **51**, 591 (1979).
([38]) A. Tagliacozzo and F. Ventriglia: this issue, p. 141.
([39]) B. I. Halperin: *Phys. Rev. Lett.*, **52**, 1583 (1984).
([40]) D. Arovas, J. R. Schrieffer and F. Wilczek: *Phys. Rev. Lett.*, **53**, 722 (1984).

where the operation of taking ψ around the loop γ has been indicated by $\hat{\gamma}\cdot\psi$. It turns out that, as has been extensively discussed in the literature (8-10,19-23,32,33,36), $a(\gamma)$ must be constant on homotopy classes of loops: $a(\gamma) = a([\gamma])$, with $[\gamma]$ the equivalence class of loops homotopic to γ, and consistent with the product of equivalence classes in π_1:

$$a([\gamma])\cdot a([\gamma']) = a([\gamma\cdot\gamma']), \tag{2.2}$$

where $\gamma\cdot\gamma'$ denotes the standard product loop (16), *i.e.* $a([\gamma])$ is a *character* of π_1. Moreover,

$$a([\gamma]^{-1} \equiv a([\gamma^{-1}]) \equiv a([\gamma])^{-1} \equiv a([\gamma])^{*}. \tag{2.3}$$

Two main consequences can be drawn at once from this general situation, and namely:

i) Let g be the element of π_1 labelling the equivalence class $[\gamma]$. Writing then directly $a(g)$ for $a([\gamma])$, we have

$$a(g\cdot h\cdot g^{-1}\cdot h^{-1}) \equiv a(g)\cdot a(h)\cdot a(g)^{*}\cdot a(h)^{*} \equiv 1 \qquad \forall g, h \in \pi_1(\boldsymbol{Q}), \tag{2.4}$$

i.e. $a \equiv 1$ on the commutator subgroup. Hence, $a(\cdot)$ is actually a homomorphism from the quotient of π_1 by the commutator subgroup into $\boldsymbol{U}(1)$. But

$$\pi_1/[\pi_1, \pi_1] = \boldsymbol{H}_1(\boldsymbol{Q}, \boldsymbol{Z}), \tag{2.5}$$

where the r.h.s. denotes the first homology group of $\boldsymbol{Q}$ with integer coefficients (also called the «abelianized» of π_1 (41)). Therefore,

$$a(\cdot) \in \mathrm{Hom}\,(\boldsymbol{H}_1(\boldsymbol{Q}, \boldsymbol{Z}), \boldsymbol{U}(1)). \tag{2.6}$$

This known result is of an enormous practical advantage, insofar as calculation of homology groups is, as a rule, much more easier than that of homotopy groups, and so is the classification of characters. The fact that it is only the abelianized π_1 which plays, so to speak, an active role, also explains why, when π_1 is perfect (or $\boldsymbol{Q}$ is «fundamentally perfect» (19)), there are no quantization ambiguities.

ii) Once a character has been found to act in eq. (2.1), consistency with the superposition principle requires $a(\cdot)$ *to be the same for all wave functions*. Wave functions which, when transported along loops, transform according to different characters cannot be linearly superposed. The Hilbert space of states breaks then down into a direct sum of «*superselection sectors*», labelled by the elements

(41) S. T. Hu: *Homotopy Theory* (Academic Press, New York, N.Y., 1959).

of Hom $(\boldsymbol{H}_1(\boldsymbol{Q},\boldsymbol{Z}),\ \boldsymbol{U}(1))$. In a slightly different but equivalent way, one can argue [9] as follows: all the physical observables must be invariant under transport along closed loops (*i.e.* they must be invariant under the action of π_1). On the other hand, the above action defines (cf. eq. (2.1)) as many unitary operators as there are distinct elements in Hom $(\boldsymbol{H}_1(\boldsymbol{Q},\boldsymbol{Z}),\ \boldsymbol{U}(1))$. As (except from trivial cases) the latter are not all multiples of the identity, and they commute with the set of the observables, by Schur's lemma the observables cannot act irreducibly on the Hilbert space of states, which will then split into a direct sum of spaces labelled by the unitary, one-dimensional representations of π_1, *i.e.* again by the characters, as before.

Whenever the r.h.s. of (2.6) is not trivial, we obtain *inequivalent quantum-mechanical descriptions of the same classical dynamical system in each superselection sector*. Going back for a moment to the examples discussed in sect. **1**, we have

i) If $\boldsymbol{Q}=\boldsymbol{R}-(0)$, $\pi_1(\boldsymbol{Q})=\boldsymbol{Z}\equiv \boldsymbol{H}_1(\boldsymbol{Q})$, and the characters are of the form

$$a(n)=\exp[in\theta]\,,\qquad n\in\boldsymbol{Z},\ \ \theta\in[0,2\pi)\,. \tag{2.7}$$

The inequivalent quantizations are then labelled by a «θ-angle» much as in the case of non-Abelian gauge theories [34] or superconducting junctions [38].

ii) For identical particles in $\boldsymbol{R}^d$, $d\geqslant 3$, $\pi_1(\boldsymbol{Q})=\boldsymbol{S}_n$ which is non-Abelian. However, $\boldsymbol{H}_1(\boldsymbol{S}_n)=\boldsymbol{S}_n/[\boldsymbol{S}_n,\boldsymbol{S}_n]=\boldsymbol{Z}_2$, just as in the case of the rigid rotator, and one has only two distinct characters, namely

$$a_1(g)\equiv 1 \qquad\text{and}\qquad a_2(g)=\pm 1 \tag{2.8}$$

according to whether g is an even or odd permutation leading unambiguously to Bose or Fermi quantization [22].

According to eq. (2.1), in a generic superselection sector we are dealing with *multivalued* wave functions. A natural way to accommodate this situation is [23] to describe quantum mechanics not on $\boldsymbol{Q}$, but on its universal covering space [16] $\overline{\boldsymbol{Q}}$, which is simply connected, supplementing such a description with appropriate conditions under which the quantum-mechanical description on $\overline{\boldsymbol{Q}}$ is projectable onto $\boldsymbol{Q}$.

We recall that $\overline{\boldsymbol{Q}}$ has the structure of a principal fiber bundle with structure group π_1. Denoting by $\pi\colon\overline{\boldsymbol{Q}}\to\boldsymbol{Q}$ the canonical projection associated with the bundle structure of $\overline{\boldsymbol{Q}}$, and hence by $\pi^{-1}(q)$ the fiber over $q\in\boldsymbol{Q}$, π_1 will act transitively and freely on fibers:

$$\overline{q},\overline{q}'\in\pi^{-1}(q)\Leftrightarrow\exists g\in\pi_1(\boldsymbol{Q})\ \text{such that}\ \overline{q}'=g\cdot\overline{q}\,,\qquad \overline{q}'=\overline{q}\Leftrightarrow g=e\,. \tag{2.9}$$

Of course, any path joining any two points in the fiber over q will project down to a loop in $\boldsymbol{Q}$ based at q and belonging to the homotopy class of loops labelled by the element $g \in \pi_1$ defined in eq. (2.9), which is unique because the action of π_1 on fibers is free.

The observables are defined in terms of local operators (differential operators on $L_2(\boldsymbol{Q}, \boldsymbol{C}, \mathrm{d}\mu)$, *e.g.*). They can therefore be lifted without ambiguities to symmetric operators acting on the Hilbert space of square-integrable functions on $\overline{\boldsymbol{Q}}$. Finally, projectability of the quantum-mechanical description from $\overline{\boldsymbol{Q}}$ onto $\boldsymbol{Q}$ will be achieved if we require wave functions on $\overline{\boldsymbol{Q}}$ to obey «generalized Bloch» boundary conditions of the form

$$\overline{\psi}(g\,\overline{q}) = a(g)\cdot\overline{\psi}(\overline{q}) \tag{2.10}$$

with

$$a(g)\cdot a(g') = a(g\cdot g'), \qquad a(g)\,a(g)^* = 1\,; \tag{2.11}$$

this will ensure that quantities like $\overline{\psi}^*\overline{\psi}$ are constant along the fibers, and depend therefore only on the base point, *i.e.* are well-defined functions on $\boldsymbol{Q}$. We recover in this way the classification of the admissible quantum-mechanical descriptions in terms of the characters of the fundamental group, but working with single-valued wave functions on the universal covering.

We turn now to a brief analysis of the structure of the propagator. On $\overline{\boldsymbol{Q}}$, the propagator can be defined via the path integral or, equivalently, through the integral form of the Schrödinger equation:

$$\overline{\psi}(\overline{q}, t) = \int_{\overline{\boldsymbol{Q}}} \mathrm{d}\mu(\overline{q}')\,\overline{K}(\overline{q}, t; \overline{q}', t')\,\overline{\psi}(\overline{q}', t')\,, \qquad t \geqslant t'\,, \tag{2.12}$$

relating the wave function at time t' to that at time t, while the measure $\mathrm{d}\mu(\overline{q})$ is obtained by lifting in an obvious way the measure $\mathrm{d}\mu(q)$ originally given on $\boldsymbol{Q}$ to a measure on $\overline{\boldsymbol{Q}}$ invariant under the action of π_1: $\mathrm{d}\mu(g\cdot\overline{q}) = \mathrm{d}\mu(\overline{q}) = \mathrm{d}\mu(\pi(\overline{q}))$. Under these conditions, it is easy to prove the following.

Theorem. The boundary conditions (2.10) will be preserved by the dynamical evolution if the propagator on the universal covering satisfies

$$\overline{K}(g\cdot\overline{q}, t; g\cdot\overline{q}', t') = \overline{K}(\overline{q}, t; \overline{q}', t') \qquad \forall\,\overline{q}, \overline{q}' \text{ and } g\,. \tag{2.13}$$

Indeed, the measure being invariant, one finds

$$\overline{\psi}(g\cdot\overline{q}, t) = \int_{\overline{\boldsymbol{Q}}} \mathrm{d}\mu(\overline{q}')\,\overline{K}(g\cdot\overline{q}, t; g\cdot\overline{q}', t')\,\overline{\psi}(g\cdot\overline{q}', t')\,.$$

If (2.10) is satisfied at the initial time t',

$$\overline{\psi}(g\cdot\overline{q},t)=a(g)\cdot\int_{\overline{Q}}\mathrm{d}\mu(\overline{q}')\,\overline{K}(g\cdot\overline{q},t;g\cdot\overline{q}',t')\,\overline{\psi}(\overline{q}',t')\,. \tag{2.14}$$

and the theorem follows at once, with the «only if» part following from the arbitrariness of the initial wave function. □

An immediate corollary of (2.12) is

$$\overline{K}(g\cdot\overline{q},t;g'\cdot\overline{q}',t')=\overline{K}((g'^{-1}\cdot g)\cdot\overline{q},t;\overline{q}',t')\qquad\forall\overline{q},\overline{q}'\text{ and }g,g'\,. \tag{2.15}$$

The proof is straightforward and will be omitted.

If the Hamiltonian is time independent and, for simplicity, it is assumed to have a discrete and nondegenerate spectrum (both conditions can be relaxed rather easily) with eigenvalues E_n and eigenfunctions, subject to the «Bloch» conditions (2.10), $\overline{\psi}_n(\overline{q})$, then an explicit form for the propagator will be

$$\overline{K}(\overline{q},t;\overline{q}',t')=\sum_n\exp\left[-\frac{i}{\hbar}E_n(t-t')\right]\overline{\psi}_n(\overline{q})\,\overline{\psi}_n^*(\overline{q}') \tag{2.16}$$

and one easily finds (we omit the time labels when they are not necessary)

$$\overline{K}(g\cdot\overline{q},g'\cdot\overline{q}')=a(g)\,a(g')^*\,\overline{K}(\overline{q},\overline{q}')\,, \tag{2.17}$$

a more specific relation, which encompasses both (2.13) and (2.15).

In order to see how the dynamical evolution projects down onto the base space $\boldsymbol{Q}$, let us choose a fundamental domain (16) in $\overline{\boldsymbol{Q}}$ which, with a slight abuse of notation, we shall denote again as $\boldsymbol{Q}$. We can, therefore, define the wave function on $\boldsymbol{Q}$ as

$$\psi(q):=\overline{\psi}(\overline{q})\,;\qquad\pi(\overline{q})=q \tag{2.18}$$

and $\overline{q}$ ranges over the chosen fundamental domain. This amounts to choosing one branch of the wave function, which is multivalued on $\boldsymbol{Q}$.

Splitting the integration over $\overline{\boldsymbol{Q}}$ into a sum of integrals over the (possibly denumberably infinite) $[\pi_1]:=\dim(\pi_1)$ copies of the fundamental domain $\boldsymbol{Q}$, we obtain

$$\psi(q)=\int_{\overline{Q}}\mathrm{d}\mu(\overline{q}')\,\overline{K}(q,t;\overline{q}',t')\,\overline{\psi}(\overline{q}',t')=\\ =\sum_{g\in\pi_1}\int_{Q}\mathrm{d}\mu(q')\,\overline{K}(q,t;gq',t')\,a(g)\,\psi(q',t') \tag{2.19}$$

and, using (2.15),

$$\psi(q)=\int_{Q}\mathrm{d}\mu(q')\,K(q,t;q',t')\,\psi(q',t')\,, \tag{2.20}$$

where

$$K(q,t;q',t') := \sum_{g\in\pi_1} a(g)\,\overline{K}(g^{-1}\cdot q, t; q', t'), \tag{2.21}$$

or, as

$$\chi(g) := a(g^{-1}) \equiv a(g)^*$$

is also a character

$$K(q,t;q',t') = \sum_{g\in\pi_1} \chi(g)\,K_g(q,t;q',t'), \tag{2.22}$$

where

$$K_g(q,t;q',t') := \overline{K}(g\cdot q, t; q', t'). \tag{2.23}$$

Fixing a homotopy mesh as discussed in sect. **1**, K_g can be identified, «modulo» a g-independent phase factor, with the path integral on the base manifold $\boldsymbol{Q}$ restricted to the homotopy class of paths labelled by the element $g \in \pi_1$. Equation (2.22) constitutes then the proof of the theorem of Schulman, Laidlaw, Morette-de Witt and Dowker.

Note that eq. (2.17) has not been used in the proof of eq. (2.22). On the other hand, if we take, say, q around a loop γ in $\boldsymbol{Q}$ in such a way that all paths from q' to q labelled by g go into those labelled by $h\cdot g$ for some $h\in\pi_1$, then, calling $K(\hat\gamma q, q')$ the resulting propagator (again, we suppress time labels here), we have

$$K(\hat\gamma q, q') = \sum_g \chi(g)\,K_{hg}(q,q') = \chi(h^{-1})\sum_g \chi(h\cdot g)\,K_{hg}(q,q'), \tag{2.24}$$

i.e. (cf. also eq. (2.21))

$$K(\hat\gamma q, q') = \chi(h)^*\,K(q,q'). \tag{2.25}$$

Taking *both* q and q' along the same loop does not alter the homotopy class to which the various paths from q' to q belong. This implies that taking q' along γ must change the homotopy class labelled by g into that labelled by $h^{-1}\cdot g$, and we obtain

$$K(q, \hat\gamma q') = \chi(h)\,K(q,q'). \tag{2.25$'$}$$

Together, the last two equations constitute a proof of (2.17), obtained without making use of the decomposition (2.16), and for the propagator on the base manifold $\boldsymbol{Q}$.

The choice of the character χ in (2.22) (or, equivalently, of the boundary conditions on wave functions on the universal covering) defines the superselection sector in which the quantum-mechanical description is being given. The «trivial» sector will be defined by the choice $\chi(g) \equiv 1$, $\forall g \in \pi_1(\boldsymbol{Q})$.

As a final remark to this section, let us note that, while in a generic sector the propagator is a multivalued function of its initial and end points (cf. eqs. (2.25)-(25′)), and it is invariant, according to (2.13), only if *both* points are taken around the same loop, in the «trivial» sector it is *separately* invariant, *i.e.* it is a *single-valued* function under independent variations (around loops) of both its initial and end points.

3. – Fiber bundle classification of inequivalent quantizations.

The classification of inequivalent quantizations using the characters of $\pi_1(\boldsymbol{Q})$ presented in sect. **2** has a bundle interpretation which provides in fact an alternative proof of the same classification scheme. The clue is to relate the «Feynman factor» $\exp[(i/\hbar)S[\gamma]]$ to parallel transport on a suitable $U(1)$ bundle with connection [6,14].

According to the discussion of sect. **1**, if a global Lagrangian exists on what is called by Souriau [6] the «evolution space»:

$$E := T\boldsymbol{Q} \times \boldsymbol{R}, \tag{3.1}$$

the action is given, in terms of the Lagrangian one-form $\theta_{\mathscr{L}}$ defined in eq. (1.5), by

$$S[\gamma] = \int_{\tilde{\gamma}} \theta_{\mathscr{L}} \tag{3.2}$$

where γ is a path in $\boldsymbol{Q}$ and $\tilde{\gamma}$ the lifted path (1.7). If the Lagrangian is regular,

$$\sigma := \mathrm{d}\theta_{\mathscr{L}} \tag{3.3}$$

is a two-form of maximal rank on E. The evolution space becomes then an exact contact manifold [3] and the solutions of the Euler-Lagrange equations are the characteristic curves of σ [3,6].

In more general cases, it may happen that the dynamics is again given as above, but in terms of a closed but not exact two-form σ, again assumed to be of maximal rank. This is the case, *e.g.*, of the charge-monopole system [8,12]. If this is the case, a one-form θ satisfying (3.3) will exist only locally. The conditions to be satisfied by σ in order that θ be a Lagrangian one-form for some, again locally defined, Lagrangian, are reviewed, *e.g.*, in ref. [13]. In this kind of situation, the very definition of the classical action (3.2) becomes problematic. However, the Feynman factor can nevertheless be defined in a nonambiguous way if [6,14,15,27] the two-form $\sigma/\hbar$ defines an integer cohomology class, *i.e.* if

$$\int_S \sigma = 2\pi\hbar \times (\text{integer}) \tag{3.4}$$

for any two-cycle S in E (details are explained, *e.g.*, in ref. ([27b])). For the charge-monopole system, eq. (3.4) provides the celebrated Dirac quantization condition ([42])

$$2eg = (\text{integer}) \times \hbar \tag{3.5}$$

with e and g the electric and magnetic charges, respectively.

Condition (3.4) guarantees ([6,14,15]) the existence of a principal $U(1)$ bundle $\boldsymbol{Y}$ over E, carrying a connection ω whose curvature $\mathrm{d}\omega$ is given by

$$\mathrm{d}\omega = \frac{\sigma}{\hbar}. \tag{3.6}$$

The procedure outlined here is called «prequantization» by Konstant and Souriau, and the bundle $(\boldsymbol{Y}, \omega)$ is called a «prequantum bundle».

Let now

$$\Gamma : [0, 1] \to E \tag{3.7}$$

be a path in E. In particular, Γ could be, after reparametrization, the lifted path $\tilde{\gamma}$ discussed in sect. **1**. If $\Gamma(0) = m \in E$, then, given any point y in the fiber over m, there is a unique ([11]) *horizontal lift* Γ_h of Γ in $\boldsymbol{Y}$, defined by the conditions

$$\Gamma_h(0) = y, \qquad \omega\left(\frac{\mathrm{d}\Gamma_h}{\mathrm{d}t}\right) = 0. \tag{3.8}$$

If Γ lies in the domain of a section:

$$s : E \supset \boldsymbol{U} \to \boldsymbol{Y}; \qquad \pi \cdot s = \mathrm{Id}_E \tag{3.9}$$

with $\boldsymbol{U}$ a contractible open set on which

$$\sigma|_{\boldsymbol{U}} = \mathrm{d}\theta, \qquad \theta = s^* \omega, \tag{3.10}$$

then there exist complex numbers z_0, $z_1 \in U(1)$ such that

$$y = \Gamma_h(0) = s(m) \cdot z_0, \qquad \Gamma_h(1) = s(\Gamma(1)) \cdot z_1. \tag{3.11}$$

([42]) P. A. M. Dirac: *Proc. R. Soc. London, Ser. A*, **133**, 60 (1931).

The connection one-form can be written in $\pi^{-1}(\boldsymbol{U})$ (with π: $\boldsymbol{Y}\rightarrow E$ the canonical projection, and with some slight abuse of notation) as

(3.12)
$$\omega = \theta - i\hbar \frac{\mathrm{d}z}{z}$$

and one easily proves ([27a-c]) that

(3.13)
$$\exp\left[\frac{i}{\hbar}\int_{\Gamma}\theta\right] = \frac{z_0}{z_1}.$$

Equation (3.13) is the fundamental relation between the «Feynman factor» and the prequantum bundle $(\boldsymbol{Y}, \omega)$.

One of the main results of the geometric quantization program ([6,14,15]) is that, given a classical dynamical system described by the pair (E, σ), with E an evolution space and σ a closed two-form of maximal rank on E, the *distinct* (equivalence classes of) *prequantum bundles are in one-to-one correspondence with the characters of the fundamental group*. As discussed in sect. 2, the characters are the elements of Hom$(\boldsymbol{H}_1(\boldsymbol{Q}, \boldsymbol{Z}), U(1))$. They form themselves a (Abelian) group under «pointwise» multiplication in π_1, *i.e.* for any two characters χ and χ', the product $\chi\cdot\chi'$, defined as

(3.14)
$$(\chi\cdot\chi')(g) := \chi(g)\cdot\chi'(g)\,, \qquad g\in\pi_1\,,$$

is again a character. A general decomposition theorem ([41]) states that the most general structure of $\boldsymbol{H}_1(\boldsymbol{Q}, \boldsymbol{Z})$ is

(3.15)
$$\boldsymbol{H}_1(\boldsymbol{Q}, \boldsymbol{Z}) = b_1\boldsymbol{Z}\oplus \mathrm{Tors}\,,$$

where b_1 is the first Betti number of $\boldsymbol{Q}$ ([11]). $\boldsymbol{H}_1$ contains then b_1 factors of $\boldsymbol{Z}$ and a residual part, the «torsion» group denoted by «Tors», which is a finite group. The torsion part would be absent in the decomposition if one considered the homology with coefficients in $\boldsymbol{R}$ instead of $\boldsymbol{Z}$.

The group of characters has then as many connected components as the dimension of its torsion part, the connected component of the identity corresponding to the identity in «Tors», and a general character $\chi(g)$ will be of the form

(3.16)
$$\chi(g) = \left[\exp\left[i\sum_{k=1}^{b_1} n_k\theta_k\right]\right]\cdot\chi_j\,,$$

with $0\leqslant\theta_k\leqslant 2\pi$, and $\chi_j\in$ Hom(Tors, $\boldsymbol{U}(1)$). By deRham's first theorem ([43]), one

([43]) H. FLANDERS: *Differential Forms* (Academic Press, New York, N.Y., 1963).

can find b_1 closed one-forms A_k and constants e_k such that

$$n_k \theta_k = \frac{e_k}{\hbar c} \oint_{[\gamma]=g} A_k . \tag{3.17}$$

Remark. It is of course only the product $e_k \cdot A_k$ which is determined by deRham's theorem. We have chosen the representation (3.17) only for further convenience. If e_k is chosen to have the dimension of an electric charge, then $2\pi\hbar c/e_k$ is an «elementary fluxon» associated with the charge and A_k has the dimensions of a (curlless) vector potential.

In terms of the forms A_k (3.16) becomes then

$$\chi(g) = \left[\exp\left[\frac{i}{\hbar c} \sum_k \oint_{[\gamma]=g} A_k \right] \right] \cdot \chi_j . \tag{3.18}$$

The description of prequantum bundles resulting from the previous discussion is as follows: there exist as many topologically inequivalent bundles as there are components in the group of characters, *i.e.* elements in Tors. On a fixed bundle, the inequivalent connections correspond to the characters in a connected component, *i.e.* to the first factor in (3.18). The situation is summarized in fig. 1.

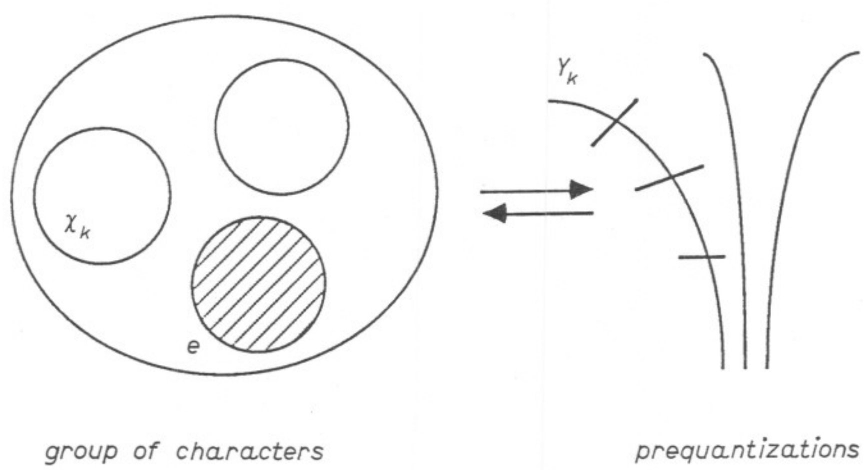

Fig. 1. – The topologically distinct bundles are labelled by the components of the group of characters. On a given bundle, the inequivalent connections are in bijection with the characters in the component.

The extreme situations correspond either to the absence of torsion, or to $b_1 = 0$. The first case occurs, *e.g.*, in the already discussed case of $\boldsymbol{Q} = \boldsymbol{R}^2 - (0)$. Here $\pi_1 = \boldsymbol{H}_1 = \boldsymbol{Z}$, and there is just one prequantum bundle, the trivial one:

$$\boldsymbol{Y} = E \times U(1) \tag{3.19}$$

and the inequivalent quantizations are labelled by the characters

$$\chi(n) = \left[\exp\left[\frac{ie}{\hbar c}\oint A\right]\right]^n, \tag{3.20}$$

where the integral of the closed one-form A is taken around a loop circling once around the origin in the positive sense. In this case, if $\theta_{\mathscr{L}}$ is the Lagrangian one-form describing the motion of a particle in the (punctured) plane, the connection form associated with the choice of A in (3.20) will be

$$\omega = \theta_{\mathscr{L}} + \frac{eA}{c} - i\hbar\frac{\mathrm{d}z}{z}. \tag{3.21}$$

Two connection one-forms will be equivalent iff they rise to the same characters, *i.e.* iff the «fluxes» $\oint A$ in (3.20) differ by integer multiples of the «elementary fluxon» $2\pi\hbar c/e$.

The other extreme is provided by $n \geq 2$ identical particles moving in $\boldsymbol{R}^d$, $d \geq 3$. As already discussed, in such a case $\boldsymbol{H}_1 \equiv \mathrm{Tors} = \boldsymbol{Z}_2$, and there will be just two inequivalent bundles, a trivial one associated with the characters of the form a_1 (cf. eq. (2.8)), and a «twisted» one, associated with those of the form a_2, the former corresponding to Bose statistics, the latter to Fermi statistics.

4. – Some examples.

In the previous sections, the general theory was illustrated by occasional discussion of some simple examples. In this one we rediscuss in a more systematic way the same examples, together with others, less elementary ones, and which can lead to interesting physical situations.

i) *The Aharonov-Bohm effect.* When $\boldsymbol{Q} = \boldsymbol{R}^2 - (\boldsymbol{0})$, the general formula (2.22) specializes to

$$K(\boldsymbol{r}, \boldsymbol{r}'; t) = \sum_{n=-\infty}^{\infty} \exp[2\pi i n\delta] K_n(\boldsymbol{r}, \boldsymbol{r}'; t), \tag{4.1}$$

where

$$\delta := \frac{\phi}{\phi_0}, \qquad \phi_0 := 2\pi\frac{\hbar c}{e}. \tag{4.2}$$

The superselection sectors are then labelled by the parameter $\delta \in [0, 1]$, or by a parameter $\phi \pmod{\phi_0}$ having the dimension of a magnetic flux. K_n is the free-particle partial kernel in the homotopy class of paths labelled by the winding number n. Introducing polar coordinates, with $\boldsymbol{r} \equiv (r, \theta)$, $\boldsymbol{r}' \equiv (r', \theta')$, K_n can be evaluated explicitly as [28]

$$(4.3)\qquad K_n(r,\theta;r',\theta';t)=$$
$$=\frac{m}{it}\exp\left[\frac{im}{2t}(r^2+r'^2)\right]\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\exp\left[ik(\theta-\theta'+2n\pi)-i|k|\frac{\pi}{2}\right]\cdot J_{|k|}\left(\frac{mrr'}{t}\right),$$

where m is the particle's mass and $J_{|k|}$ a Bessel function (see ref. [28] for details of the derivation). Note that K_n satisfies

$$(4.4)\qquad \theta\to\theta\pm 2\pi\Rightarrow K_n\to K_{n\pm1}\,,\qquad \theta'\to\theta'\pm 2\pi\Rightarrow K_n\to K_{n\mp1}\,,$$

(4.4) guarantees that the general property of eq. (2.25) is satisfied and, in particular, that the propagator will be single-valued in $\boldsymbol{r}$ and $\boldsymbol{r}'$ separately in the «trivial» sector $\delta=0$.

The *a posteriori* physical interpretation of (4.1) is of course that ϕ is the magnetic flux associated with an infinitely long, straight flux line piercing the plane at the origin, and the Aharonov-Bohm effect seems to have a natural setting in terms of multivalued (propagators and) wave functions. However, it has been shown in ref. [26] and [28] that one can give a complete description of the effect in the «trivial» sector as well. Indeed, considering the one-form

$$(4.5)\qquad \Theta=\Theta_0-\alpha\,,\qquad \alpha:=\frac{e\phi}{2\pi c}\,\mathrm{d}\theta$$

obtained by adding to the free-particle Lagrangian one-form Θ_0 the closed but not exact form α, the path integral can be performed exactly as before, and the final result is [28]

$$(4.6)\qquad K(\boldsymbol{r},\boldsymbol{r}';t)=\sum_{n=-\infty}^{\infty}\exp\left[i\delta(\theta-\theta'+2n\pi)\right]K_n(\boldsymbol{r},\boldsymbol{r}';t)$$

with δ and K_n given again by (4.2) and (4.3). By virtue of (4.4), (4.6) is single-valued and leads to the same physical predictions as (4.1).

The choice (4.5) corresponds to the addition to the action of a topological term, which has a straightforward physical interpretation, and is ineffective at the classical but not at the quantum level. A more general situation of this sort is discussed in ref. [23c], and will be reanalysed in the next section.

The case examined here corresponds to $b_1=1$. The more general case $b_1>1$ would correspond to a «multipunctured» plane. Consider, *e.g.*, the case $b_1=2$, *i.e.* a plane with two holes in it. The basic topology is that of a «figure-eight» [16], and the fundamental group is non-Abelian:

$$(4.7)\qquad \pi_1=\boldsymbol{Z}*\boldsymbol{Z},$$

the free group on two generators. However,

$$H_1 = Z \oplus Z \tag{4.8}$$

and (4.1) generalizes into

$$K(\mathbf{r}, \mathbf{r}'; t) = \sum_{n,m=-\infty}^{\infty} \exp[2\pi i(n\delta_1 + m\delta_2)] \cdot K_{n,m}(\mathbf{r}, \mathbf{r}'; t) \tag{4.9}$$

with the pairs (n, m) labelling homotopy classes, and δ_1, δ_2 defining the fluxes through the two holes in the plane. The case $\delta_1 = -\delta_2 = \delta$ corresponds to the physically interesting case of an idealized model of a solenoid closing onto itself (or of a cross-section of a toroidal solenoid, a case has been the subject of highly sophisticated experimental investigations ([44])). In such a case (we drop arguments for the sake of brevity),

$$K = \sum_{n=-\infty}^{\infty} \exp[in\delta] \cdot \tilde{K}_n, \qquad \tilde{K}_n := \sum_{m=-\infty}^{\infty} K_{m,m-n} . \tag{4.10}$$

ii) *Identical particles.* The case of identical particles moving in $\boldsymbol{R}^d$, $d \geqslant 3$, has already been discussed. We consider here other cases.

ii-*a*) *Two identical particles on a torus.* Let us consider a three-torus $\boldsymbol{T}^3$. According to the discussion of sect. 2 (see also ([22,23c,30])), the appropriate configuration space is

$$\boldsymbol{Q} = \frac{\tilde{\boldsymbol{Q}}}{\boldsymbol{Z}_2}, \qquad \tilde{\boldsymbol{Q}} = \boldsymbol{T}^3 \times \boldsymbol{T}^3 - (\text{diagonal}) . \tag{4.11}$$

The exclusion of the (three-dimensional) diagonal from $\boldsymbol{T}^3 \times \boldsymbol{T}^3$ does not change the loop topology of the space, and hence $\pi_1(\tilde{\boldsymbol{Q}}) = \boldsymbol{Z} \oplus \boldsymbol{Z}$.

$\boldsymbol{Z}_2$ acts on $\tilde{\boldsymbol{Q}}$ by interchange. Let us recall the following definition: given two groups $\boldsymbol{A}$ and $\boldsymbol{B}$, and a homomorphism $f: \boldsymbol{A} \to \mathrm{Aut}(\boldsymbol{B})$, there is a group, denoted by $\boldsymbol{G} = [\boldsymbol{B}]\boldsymbol{A}$, with elements (b, a), $b \in \boldsymbol{B}$, $a \in \boldsymbol{A}$ and multiplication table:

$$(b, a) \cdot (b', a') := (f(a')(b) \cdot b', a \cdot a') . \tag{4.12}$$

In our case, $\boldsymbol{A} = \boldsymbol{Z}_2$ with elements denoted by ± 1 (and ordinary multiplication as

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([44]) A. TONAMURA, T. MATSUDA, R. SUZUKI, A. FUKUHARA, N. OSAKEBE, H. UMEZAKI, J. ENDO, K. SHINAGAWA, Y. SUGITA and H. FUJIWARA: *Phys. Rev. Lett.*, 48, 1443 (1982); A. TONAMURA, H. UMEZAKI, T. MATSUDA, N. OSAKABE, J. ENDO and Y. SUGITA: *Phys. Rev. Lett.*, **51**, 331 (1983).

the group operation). Denoting by (m, n) an element of $\boldsymbol{Z} \oplus \boldsymbol{Z}$:

$$f(\alpha):(m, n) \to \begin{cases} (m, n) & \alpha = 1\,, \\ (n, m) & \alpha = -1 \end{cases} \tag{4.13}$$

and

$$\pi_1(\boldsymbol{Q}) = [\boldsymbol{Z} \oplus \boldsymbol{Z}]\,\boldsymbol{Z}_2 \tag{4.14}$$

with the multiplication table

$$((m, n), \alpha) \cdot ((m', n', \beta) = \begin{cases} ((m + m', n + n'), \alpha\beta)\,, & \beta = 1\,, \\ ((m' + n, n' + m), \alpha\beta)\,, & \beta = -1\,. \end{cases} \tag{4.15}$$

The first homology group turns out to be

$$\boldsymbol{H}_1(\boldsymbol{Q}, \boldsymbol{Z}) = \boldsymbol{Z} \oplus \boldsymbol{Z}_2 \tag{4.16}$$

and the characters, labelled by the pairs (n, α), are of the form

$$\chi(n, \alpha) = \left[\exp\left[\frac{ie}{\hbar c}\int A\right]\right]^n \cdot \chi_i(\alpha)\,, \tag{4.17}$$

$i = 1, 2$, where $\chi_1(\pm 1) = 1$, $\chi_2(\pm 1) = \pm 1$. We see here the simultaneous appearance of an «Aharonov-Bohm»-type situation and of the Fermi-Bose alternative deriving from the torsion part of $\boldsymbol{H}_1$.

For identical particles on a two-torus $\boldsymbol{T}^2$ the situation is more intrigued because, in this case, the exclusion of the diagonal changes the loop topology of the space, and we refer to the literature [27f,45] for details of the analysis.

ii-*b*) *Identical particles in* $\boldsymbol{R}^2$. So far, we have considered identical particles on compact manifolds. The situation is different in $\boldsymbol{R}^2$ and has been analysed, *e.g.*, in ref. [30,31] (but see also [23c,32]). In this case too the exclusion of the diagonal changes the topology, and $\boldsymbol{R}^{2n}$ − (diagonal) is no more simply connected. For example [23c], for $n = 2$, $\boldsymbol{R}^4$ − (diagonal) can be shown, via successive deformation retracts, to have the same topology as $\boldsymbol{S}^1$, and hence $\pi_1 = \boldsymbol{Z}$. Quotienting with $\boldsymbol{S}^n$ we obtain the final configuration space $\boldsymbol{Q}$, whose fundamental group turns out to be [31] the $(n-1)$-string Artin braid group [46],

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[45] T. D. IMBO and C. S. IMBO: to be published.
[46] E. ARTIN: *Ann. Math.*, 48, 101 (1947).

which is defined by $(n-1)$ generators $\sigma_1 \dots \sigma_{n-1}$, and by the relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{4.18}$$

and

$$\sigma_i \sigma_k = \sigma_k \sigma_i \qquad \forall i \neq k \pm 1 \tag{4.19}$$

(for $n = 2$, there is a single generator, and we recover $\pi_1 = \boldsymbol{Z}$). The group of characters will be generated by $\chi(\sigma_i)$, $i = 1, \dots n-1$. Setting $\chi(\sigma_1) := \exp[-i\theta_i]$, (4.19) is automatically satisfied, while (4.18) is easily seen to imply $\theta_i = \theta_{i+1}$ (mod 2π). Hence, the characters are labelled by a single angle:

$$\chi(\sigma_i) = \exp[-i\theta], \qquad 0 \leqslant \theta < 2\pi. \tag{4.20}$$

This is consistent with the fact that [23c] $\boldsymbol{H}_1(\boldsymbol{Q}, \boldsymbol{Z}) = \boldsymbol{Z}$. Note that, in this case, there is no torsion.

According to what has been said in sect. **3** (cf. eqs. (3.16)-(3.18)), characters are determined by a closed but not exact one-form. Wu [31] has shown that σ_i can be interpreted as the operation of interchange of the i-th and the $(i+1)$-th particle, and hence, if γ is a loop in the homotopy class labelled by $g \in \pi_1(\boldsymbol{Q})$, the corresponding character can be represented as

$$\chi(g) = \exp\left[-\frac{i\theta}{\pi}\int_\gamma \left(\sum_{i<j} \mathrm{d}\phi_{ij}\right)\right], \tag{4.21}$$

where ϕ_{ij} is the relative azimuthal angle between the i-th and the j-th particle. Note that:

i) Along a loop in $\boldsymbol{Q}$, ϕ_{ij} can only change by a multiple of π. So $\chi(g) = \exp[-in\theta]$ for some $n \in \boldsymbol{Z}$, the familiar form of the characters when $\pi_1 = \boldsymbol{Z}$.

ii) A loop corresponding to an interchange of particles i and j has $\Delta\phi_{ij} = \pm\pi$, all the others being zero, and $\chi(g) = \exp[\mp i\theta]$. Under particle interchange, wave functions change by the above phase factor. The various inequivalent quantizations are labelled by the angle θ and the particles obey anomalous «θ-statistics» which interpolate between Bose ($\theta = 0$) and Fermi ($\theta = \pi$) statistics. They are «anyons» in the sense of the word first introduced by Wilczek [47] in a slightly different but related context.

To conclude this section, we recall that «θ (or "anyon")-statistics» seem to play

(47) F. WILCZEK: *Phys. Rev. Lett.*, **49**, 957 (1982).

an important role in the theory of the fractionally quantized Hall effect [48]. Indeed, it has been argued [39,40] that elementary excitations above Laughlin's variational ground state [49], which is appropriate for the description of a 2D strongly correlated electron gas in an external magnetic field at filling (the fraction of occupied Landau levels) $\nu = 1/m$ (m an odd integer) carry a fractional charge: $e^* = \nu e$, and obey «anyon» statistics with $\theta = \pi\nu$. This seems to be an interesting example in which, among all the kinematically possible inequivalent quantizations, a specific one is chosen, the choice being dictated essentially [49] by energetical, *i.e.* dynamical, reasons.

5. – Generalizations and discussion.

Up to now we have been discussing «scalar» quantum mechanics, *i.e.* the case in which state vectors can be represented as complex-valued (square-integrable) functions. As emphasized in previous sections, the main result concerning the scalar case is that inequivalent quantizations are classified by $\mathrm{Hom}(\boldsymbol{H}_1(\boldsymbol{Q}, \boldsymbol{Z}), \boldsymbol{U}(1))$, *i.e.* by the group of characters of the fundamental group of the configuration space $\boldsymbol{Q}$. The group of characters is trivial (and quantization is unique) iff $\pi_1(\boldsymbol{Q})$ is a perfect group [20]. To investigate wheter there are physically relevant examples of nontrivial configuration spaces having nonetheless perfect fundamental groups is in itself an interesting programme. One such example in scalar (*i.e.* $\boldsymbol{U}(1)$) quantum gravity has been exhibited in ref. [19].

A generalization of the above approach occurs [9,10,32] when wave functions take values not in $\boldsymbol{C}$ but in $\boldsymbol{C}^N$, $N > 1$. Before embarking into the discussion of specific examples, let us briefly outline how the scheme of the previous sections generalizes to the case considered here. The wave function will be now a vector-valued complex function:

$$\Psi(q) \equiv (\psi_1(q), \ldots, \psi_N(q)), \qquad \psi_i \in \boldsymbol{L}_2(\boldsymbol{Q}, \boldsymbol{C}, \mathrm{d}\mu) \tag{5.1}$$

with the norm defined by

$$\|\Psi\|^2 := \int_{\boldsymbol{Q}} \mathrm{d}\mu(q)\, \Psi^\dagger(q)\, \Psi(q) = \sum_{i=1}^{N} \|\psi_i\|^2 . \tag{5.2}$$

The quantity carrying the physical information will be now $\Psi^\dagger(q)\,\Psi(q)$ and this will be requested to be a well-defined, single-valued function on $\boldsymbol{Q}$. It follows that Ψ and $\boldsymbol{U}\cdot\Psi$, with $\boldsymbol{U} \in \boldsymbol{U}(N)$ an $N \times N$ unitary matrix, represent the same

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(48) J. S. PRANGE and R. GIRVIN (Editors): *The Quantum Hall Effect* (Springer, Berlin, 1987).

(49) R. B. LAUGHLIN: *Phys. Rev. Lett.*, **50**, 1395 (1983) and *Physica B*, **126**, 254 (1984).

physical quantum state. Paraphrasing the discussion of sect. **1**, quantum mechanics should now be formulated in general considering wave functions as sections of a $\boldsymbol{U}(N)$ bundle over $\boldsymbol{Q}$.

If $\boldsymbol{Q}$ is topologically nontrivial, transporting wave functions along closed loops will lead to the following generalization of eq. (2.1):

$$(\hat{\gamma}\Psi)(q) = U([\gamma])\,\Psi(q) \tag{5.3}$$

with U a unitary operator on $\boldsymbol{C}^N$ depending only on the homotopy class of paths to which the loop γ belongs, or, more explicitly,

$$(\hat{\gamma}\Psi)_i(q) = U_{ij}([\gamma])\,\psi_j(q) \tag{5.4}$$

and, moreover,

$$U([\gamma])\cdot U([\gamma']) = U([\gamma]\cdot[\gamma'])\,, \tag{5.5}$$

i.e. the «loop operators» $\hat{\gamma}$ must give rise, via the operators $U([\gamma])$, to a unitary representation of the fundamental group. Wave function superposition will be allowed only among state vectors (or wave functions) which transform according to the same representation, and the total Hilbert space $\mathcal{H}$ of states will break again into superselection sectors, *i.e.* into a direct sum of subspaces $\{\mathcal{H}_\rho\}$, with $\mathcal{H}_\rho$ containing only states which transform according to a fixed representation ρ of π_1. If ρ is reducible (a case which did not occur before), then $\mathcal{H}_\rho$ will break up further into a direct sum of subspaces transforming according to the irreducible components of ρ. The superselection sectors into which the Hilbert space $\mathcal{H}$ breaks are thus labelled by the irreducible unitary representations (IURs) of $\pi_1(\boldsymbol{Q})$.

As to the propagator, it will now be a matrix-valued function of its initial and end points, *i.e.*

$$\psi_i(q,t) = \sum_j \int_{\boldsymbol{Q}} \mathrm{d}\mu(q')\,K_{ij}(q,t;q',t')\,\psi_j(q',t') \tag{5.6}$$

and, analysing again its structure on the universal covering space, the generalization of eqs. (2.13) and (2.17) will be (matrix multiplication being understood)

$$K(g\cdot\overline{q}, g'\cdot\overline{q}') = U(g)\cdot K(\overline{q},\overline{q}')\cdot U^\dagger(g')\,. \tag{5.7}$$

The theorem of Schulman, Laidlaw, Morette-deWitt and Dowker generalizes then to the present case in an obvious way, with the unitary operator $U(g)$ replacing the character of eq. (1.20), and matrix multiplication replacing ordinary multiplication.

With reference to the discussion of sect. 3, and mainly to eq. (3.18) and the discussion following it, we have seen that, in the scalar case and in each one of the topologically nonequivalent bundles defined there, every nonequivalent quantization was associated with one or more closed one-forms. In the notation of sect. 3 the one-form

$$\omega = \frac{1}{c}\sum_k e_k A_k - i\hbar\frac{\mathrm{d}z}{z} \tag{5.8}$$

can also be seen as defining a *flat* ($\mathrm{d}\omega = 0$) connection on each $U(1)$ bundle over $\boldsymbol{Q}$. Flatness of the connection (*i.e.* closure of the A_k's) is the ultimate responsible for the A_k's generating the (nontorsion part of) characters of the fundamental group. A similar situation occurs in the more general case considered here, namely, it can be shown [32a,36] that the requirements that the unitary operators U realize a representation of π_1 is equivalent to the requirement that the $U(N)$ bundle over $\boldsymbol{Q}$ can be equipped with a flat connection. That every $U(N)$ bundle over a smooth manifold $\boldsymbol{Q}$ does indeed possess a naturally defined flat connection has been proved by Milnor [50], and his proof ensures consistency of the scheme outlined here.

We now discuss some examples to which the generalizations discussed here apply:

i) In the already-discussed case $n \geqslant 2$ particles moving in $\boldsymbol{R}^d$, $d \geqslant 3$, $\pi_1(\boldsymbol{Q}) = \boldsymbol{S}_n$. The number of IURs of $\boldsymbol{S}_n$ is equal to the number of partitions of n, denoted by $p(n)$. For example, $p(n) = 2$ for $n = 2$, $p(n) = 3$ for $n = 3$, and $p(n)$ grows rapidly with n for higher n's. So, for $n \geqslant 3$ there are, besides the two scalar representations corresponding to Bose and Fermi statistics, higher-dimensional IURs representing generalizations of the latter to nonscalar parastatistics [32,51].

ii) An asymmetric rigid rotator (which is also used as a model in the study of excitations of strongly deformed nuclei) can be described [9,10,32a] by the three distinct moments of inertia I_i, $i = 1, 2, 3$, along the principal axes. The configuration space $\boldsymbol{Q}$ is then the space of real symmetric 3×3 matrices with distinct eigenvalues equal to the moments of inertia of the rotator. Any such matrix can be written in the form

$$\boldsymbol{T} = \mathscr{R} \cdot \boldsymbol{T}_0 \cdot \mathscr{R}^{-1} \tag{5.9}$$

with $\boldsymbol{T}_0 = \mathrm{diag}\{I_1, I_2, I_3\}$ and $\mathscr{R}$ a matrix in the faithful representation of $\boldsymbol{SO}(3)$.

([50]) J. Milnor: *Commun. Math. Helv.*, **32**, 215 (1957).

([51]) *a*) F. J. Bloore, I. Bratley and J. M. Selig: *J. Phys. A*, **16**, 729 (1983); *b*) R. D. Sorkin: in *Topological Properties and Global Structure of Space-Time*, edited by P. G. Bergmann and V. De Sabbata (Plenum Press, New York, N.Y., 1986).

The configuration space is then the orbit of $\boldsymbol{T}_0$ under the action of $\boldsymbol{SO}(3)$. Noting that the l.h.s. of (5.9) is invariant under the substitution $\mathscr{R} \to \mathscr{R} \cdot \mathscr{R}_i(\pi)$, $i = 1, 2, 3$, with $\mathscr{R}_i(\pi)$ the matrix representing a rotation of π around the i-th axis (*i.e.* $\mathscr{R}_1(\pi) = \operatorname{diag}\{1, -1, -1\}$ and so on by cyclic permutation), $\boldsymbol{Q}$ will be the space of cosets of $\boldsymbol{SO}(3)$ with respect to the group of four elements:

$$\boldsymbol{D}_2 = \{\boldsymbol{I}, \mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3\}. \tag{5.10}$$

Hence $\boldsymbol{Q} = \boldsymbol{SO}(3)/\boldsymbol{H}$. But it can be shown (9) that $\boldsymbol{SO}(3)/\boldsymbol{D}_2 \simeq \boldsymbol{SU}(2)/\boldsymbol{Q}_8$, where $\boldsymbol{Q}_8$ is the group of quaternions, and hence, $\boldsymbol{SU}(2)$ being simply connected,

$$\pi_1(\boldsymbol{Q}) = \pi_1(\boldsymbol{SU}(2)/\boldsymbol{Q}_8) \simeq \boldsymbol{Q}_8. \tag{5.11}$$

There are five IURs of $\boldsymbol{Q}_8$, four of which are one-dimensional (reflecting the fact that $\boldsymbol{H}_1(\boldsymbol{Q}_8, \boldsymbol{Z}) = \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$), while the fifth is two-dimensional, allowing then for a nonscalar quantization. For more details, we refer to ref. (9,10,32a).

Intuitively, nonscalar quantum mechanics seems to correspond to some kind of internal symmetry of the system. The meaning of this statement has been made more precise in ref. (9) (but see also ref. (32)), where it has been shown that, associated with every UIR ρ of $\pi_1(\boldsymbol{Q})$, there are as many linearly independent states localized at every point $q \in \boldsymbol{Q}$ (the analogue of the states $|q\rangle$ of nonrelativistic scalar quantum mechanics considered in sect. **1**) as $\dim(\rho)$, which transform among themselves under the action of ρ.

We conclude this section by going back to the case of scalar quantum mechanics, to reconsider and generalize a situation already found in sect. 4 in the analysis of the Aharonov-Bohm effect.

With reference to what has been pictorially depicted in fig. 1 and to the discussion of sect. 3, on a given prequantum bundle (labelled by an element of the torsion group) to a given element of $\pi_1(\boldsymbol{Q})$ is associated a character of the form given in eq. (3.18). Omitting here the factor χ_i, common to all the inequivalent quantizations in the same component of the character group, the character can be represented as

$$\chi(g) = \exp\left[\frac{i}{\hbar c}\int_{L(\gamma)} \boldsymbol{A}\right], \tag{5.12}$$

where $L(\gamma)$ is the loop based at the fiducial point q_0 defined in eq. (1.19) and $\boldsymbol{A}$ is the sum of the one-forms appearing in (3.18). By defining

$$\xi_0(q) := \exp\left[-\frac{i}{\hbar c}\int_{\gamma_0(q)} \boldsymbol{A}\right], \tag{5.13}$$

one finds at once

$$\chi(g) \equiv \xi_0(q) \cdot \xi_0^*(q) \cdot \exp\left[\frac{i}{\hbar c}\int_\gamma \boldsymbol{A}\right]. \tag{5.14}$$

The suffix «0» as been inserted here to stress the fact that prefactors such as $\xi_0(q)$ are functions of both the point q *and* the path γ_0 in the chosen homotopy mesh, and not simply functions of q which could be disposed of by redefining the Lagrangian via the addition of a (globally defined) total time derivative.

It follows that the Feynman propagator can be written as

$$K(q, q') = \xi_0(q) \cdot \xi_0^*(q') \int \mathscr{D}[\gamma] \exp\left[\frac{i}{\hbar}\tilde{S}[\gamma]\right], \tag{5.15}$$

where now the path integral on the r.h.s. has to be evaluated *in the «trivial» homotophy sector, but with a modified action*, namely

$$\tilde{S}[\gamma] := \int_{\tilde{\gamma}} \tilde{\theta}, \qquad \tilde{\theta} := \theta_{\mathscr{L}} + \frac{1}{c}\boldsymbol{A}. \tag{5.16}$$

In other words, within each class of topologically nonequivalent bundles, one can always get rid of the multivaluedness of the wave functions (or reduce the calculation of the path integral to that in the «trivial» sector) at the expenses of introducing *appropriate topological terms* into the action. Let us stress that the path integral on the r.h.s. of (5.15) is (separately) single-valued in q and q', and that all the multivaluedness has thus been transferred into the prefactors.

The general situation depicted here is well exemplified by the example of the Aharonov-Bohm effect and also by that of identical particles in $\boldsymbol{R}^2$, both discussed in sect. 4 (see also ref. (23c)).

The full consequences of quantization in nonsimply connected spaces are only beginning to be investigated (though a large amount of literature is by now already available). Though the case of scalar quantization (*i.e.* essentially the case of an Abelian fundamental group) appears to be rather well understood, the non-Abelian case seems to show up features which are sufficiently novel and interesting to be worth investigating. We hope to return on all these topics in the near future.

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● RIASSUNTO

In questo lavoro si analizzano nuovi aspetti che si evidenziano nei processi di quantizzazione di un sistema dinamico in uno spazio configurazionale multiplamente connesso. Dopo una presentazione del problema sulla base del metodo dell'integrale di cammino, si mostra come sia possibile ottenere una classificazione del tipo «fiber bundle» delle quantizzazioni inequivalenti associate a spazi non semplicemente connessi. Si discutono vari esempi e la generalizzazione al caso in cui, a causa di simmetrie interne, un sistema può ammettere quantizzazioni non scalari.

Неэквивалентные квантования в многосвязных пространствах.

Резюме (*). — В этой статье анализируются норые особенности, которые проявляются в процессе квантования динамической системы на многосвязном конфигурационном пространстве. Заново обсуждается подход к этой проблеме, основанный на интегрировании по траекториям, затем мы показываем, как можно описать классификацию неэквивалентных квантований, ассоциированных с многосвязными пространствами. Мы обсуждаем различные примеры и обобщение на случай, в котором, вследствие внутренних симметрий, система может допускать нескалярные квантования.

(*) *Переведено редакцией.*