The theory of screws: A new geometric representation for the group SU(1,1)

R. Simon[a] and N. Mukunda[b]
Center for Theoretical Studies, Indian Institute of Science, Bangalore 560 012, India
E. C. G. Sudarshan[a]
Department of Physics, University of Texas at Austin, Austin, Texas 78712

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Hamilton's theory of turns, which gives a geometrical description of the elements and structure of the compact group SU(2), is generalized to a theory of screws for the noncompact group SU(1,1). Group elements are pictured as geometric objects in a three-dimensional Minkowski space, and the composition law is reduced to a geometric operation on them. A new classification of elements of SU(1,1), leading to an interesting structural result about the group manifold, is introduced.

I. INTRODUCTION

The unitary unimodular group SU(2) is the simplest example of a non-Abelian compact Lie group with a simple Lie algebra. It is of fundamental significance in quantum mechanical problems, being basic to the quantum theory of angular momentum as the covering group of the rotation group SO(3).[1] It also plays an important role in theories of internal symmetry for nuclear and particle physics; and it is of significance in polarization optics as well.[2] In the context of angular momentum theory, it is often convenient to parametrize the elements of SU(2) by Euler angles, which makes the irreducible representation matrices in a suitable basis easy to deal with. The group composition law, however, is rather cumbersome with this parametrization. On the other hand, the use of homogeneous Euler parameters simplifies the expressions for group multiplication to some extent, but involves the use of nonindependent parameters.

All these are of course algebraic ways of describing the elements and the composition law of SU(2). It is remarkable, but unfortunately not too well known, that as long ago as 1853 Hamilton had invented a geometrical or pictorial way of representing SU(2) elements and their multiplication, which is extremely elegant and gives one a direct and vivid grasp of the structure of SU(2). This is the so-called method of turns.[3] To appreciate Hamilton's method, let us first recall the much simpler case of the Abelian group of translations in Euclidean three-dimensional space. Each translation is representable as a vector in space, only the direction and magnitude being significant, and the location irrelevant. The composition of two translations is given by the head-to-tail parallelogram rule of vector addition; and taking the inverse amounts to reversing direction. In Hamilton's theory of turns, we have a generalization of such a picture from the Abelian translation group to the non-Abelian SU(2). Instead of vectors in space, we deal with directed great circle arcs, of length $\leqslant\pi$, on a unit sphere S^2 in a Euclidean three-dimensional space. Two such arcs are deemed equivalent if by sliding one along its great circle it can be made to coincide with the other. Equivalence classes of such arcs are called turns, and elements of SU(2) can be represented by them. Perhaps the most remarkable feature is that the noncommutative multiplication law of SU(2) can be translated into the language of turns thus: given two elements of SU(2), slide the corresponding representative great circle arcs on their respective great circles till the head of the first coincides with the tail of the second. Then the product element is represented by the turn determined by the great circle arc from the tail of the first to the head of the second arc. Inverses of elements go into reversed turns. A detailed account of Hamilton's work can be found in the monograph of Biedenharn and Louck.[4]

The purpose of this paper is to show that Hamilton's method can be generalized from the compact SU(2) to the noncompact group SU(1,1) in a very interesting way.[5] This group is the simplest non-Abelian noncompact Lie group with a simple Lie algebra, and shares with SU(2) a common complex extension. Like SU(2), SU(1,1) too is of great importance in many physical problems. Thus one may mention the theory of axially symmetric optical systems in first-order Fourier optics[6]; the group $Sl(2,R)$ of real linear canonical transformations in one pair of canonical variables, which is isomorphic to SU(1,1); and the group of Bogoliubov transformations on one creation-annihilation operator pair, relevant for squeezed states.[7]

We shall use the term "screws" for SU(1,1) in place of the turns of SU(2). In analyzing the geometrical properties of screws and the description of the SU(1,1) group structure using them, some important differences as compared to the SU(2) case will be evident. While the carrying over of Hamilton's ideas from SU(2) to SU(1,1) is thus not trivial, it is gratifying that it can in fact be done.

The material of this paper is arranged as follows. In Sec. II we give a brief review of the method of turns for SU(2). Section III develops the method of screws for SU(1,1) in detail, paying special attention to those features that distinguish it from SU(2) but at the same time guided by the SU(2) case. The concluding section (Sec. IV) collects some pertinent remarks and points out some possible applications of our new method.

II. REVIEW OF TURNS FOR SU(2)

In this section we review very briefly the theory of turns for SU(2), in a form suitable for the intended extension to

[a] Present address: The Institute of Mathematical Sciences, C.I.T. Campus, Madras 600 113, India.
[b] Jawaharlal Nehru Fellow.

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SU(1,1). While the content is essentially the same as in the account given in Ref. 4, it is expressed in a form convenient for our purposes.

As is well known, any matrix u in the defining representation of SU(2) [any $u \in \mathrm{SU}(2)$] can be written in terms of homogeneous Euler parameters and Pauli matrices as

$$u = a_0 - i\mathbf{a}\cdot\boldsymbol{\sigma} \tag{2.1}$$

(the unit matrix accompanying a_0 is omitted), where a_0 and $\mathbf{a}$ are a real scalar and real Euclidean three-vector constrained by

$$a^2 + \mathbf{a}\cdot\mathbf{a} = 1. \tag{2.2}$$

In this way, elements of SU(2) correspond one-to-one to points on S^3. The constraint (2.2) suggests that we choose any two unit vectors $\mathbf{n}, \mathbf{n}' \in S^2$ and set

$$a_0 = \mathbf{n}\cdot\mathbf{n}', \quad \mathbf{a} = \mathbf{n}\wedge\mathbf{n}'; \tag{2.3}$$

for then, if $\theta/2$ is the angle between $\mathbf{n}$ and $\mathbf{n}'$, and $\hat{a}$ is the unit vector along $\mathbf{n}\wedge\mathbf{n}'$,

$$a_0 = \cos\theta/2, \quad \mathbf{a} = \hat{a}\sin\theta/2, \tag{2.4}$$

and condition (2.2) is obviously satisfied.

We are thus led to define, for any $\mathbf{n}, \mathbf{n}' \in S^2$, the following SU(2) element $A(\mathbf{n},\mathbf{n}')$:

$$A(\mathbf{n},\mathbf{n}') = \mathbf{n}\cdot\mathbf{n}' - i\mathbf{n}\wedge\mathbf{n}'\cdot\boldsymbol{\sigma}. \tag{2.5}$$

One can easily convince oneself that for any a_0, $\mathbf{a}$ obeying Eq. (2.2), choices of $\mathbf{n}$, $\mathbf{n}'$ can certainly be made so that Eq. (2.3) will be valid. Thus every $u \in \mathrm{SU}(2)$ is obtained by making all possible choices of $\mathbf{n}$ and $\mathbf{n}'$ in $A(\mathbf{n},\mathbf{n}')$.

The geometrical meaning of the element $A(\mathbf{n},\mathbf{n}')$ is clarified by computing the element $R(A(\mathbf{n},\mathbf{n}')) \in \mathrm{SO}(3)$ that corresponds to it under the $\mathrm{SU}(2)\to\mathrm{SO}(3)$ homomorphism. We find:

$$\begin{aligned} &A(\mathbf{n},\mathbf{n}')\mathbf{n}\cdot\boldsymbol{\sigma}A(\mathbf{n},\mathbf{n}')^{-1} = \mathbf{n}''\cdot\boldsymbol{\sigma},\\ &\mathbf{n}'' = 2\mathbf{n}\cdot\mathbf{n}'\mathbf{n}' - \mathbf{n},\\ &\mathbf{n}\cdot\mathbf{n}' = \mathbf{n}'\cdot\mathbf{n}'', \quad \mathbf{n}\wedge\mathbf{n}' = \mathbf{n}'\wedge\mathbf{n}''. \end{aligned} \tag{2.6}$$

Thus $R(A(\mathbf{n},\mathbf{n}'))$ is a right-handed rotation about $\mathbf{n}\wedge\mathbf{n}'$ as the axis with *twice* the angle ($\leqslant\pi$) enclosed between $\mathbf{n}$ and $\mathbf{n}'$. In particular, $R(A(\mathbf{n},\mathbf{n}'))$ *does not* rotate $\mathbf{n}$ into $\mathbf{n}'$ but overshoots it in the plane of $\mathbf{n}$ and $\mathbf{n}'$ to $\mathbf{n}''$. On $\mathbf{n}'$ the effect is given by

$$A(\mathbf{n},\mathbf{n}')\mathbf{n}'\cdot\boldsymbol{\sigma}A(\mathbf{n},\mathbf{n}')^{-1} = (2\mathbf{n}'\cdot\mathbf{n}''\mathbf{n}'' - \mathbf{n}')\cdot\boldsymbol{\sigma}, \tag{2.7}$$

$\mathbf{n}''$ being as in Eq. (2.6).

The construction of $A(\mathbf{n},\mathbf{n}')$ enjoys the following properties:

$$A(\mathbf{n},\mathbf{n}')^{-1} = A(\mathbf{n}',\mathbf{n}), \tag{2.8a}$$

$$A(\mathbf{n}',\mathbf{n}'')A(\mathbf{n},\mathbf{n}') = A(\mathbf{n},\mathbf{n}''). \tag{2.8b}$$

(Here $\mathbf{n}$, $\mathbf{n}'$, and $\mathbf{n}''$ are independently chosen points on S^2.) In addition, $A(\mathbf{n},\mathbf{n}')$ is unchanged if both $\mathbf{n}$ and $\mathbf{n}'$ are subjected to a common rotation about $\mathbf{n}\wedge\mathbf{n}'$. This SO(2) invariance property motivates the following equivalence relation: joining $\mathbf{n}$ and $\mathbf{n}'$ by a great circle arc of length $\leqslant\pi$, this arc is equivalent to all other arcs obtained by sliding the given arc on its great circle. An equivalence class of arcs is called a turn.

Based on Eqs. (2.8), the SU(2) group operations can immediately be given a geometrical description. Each $u \in \mathrm{SU}(2)$ (other than $u = \pm\mathbb{1}$) corresponds to a unique turn. For $u = \mathbb{1}$ we take the null turn, i.e., any $\mathbf{n} = \mathbf{n}' \in S^2$. For $u = -\mathbb{1}$, *any* great semicircle will do, and they are all deemed equivalent. The inverse of $u = A(\mathbf{n},\mathbf{n}')$ corresponds, by Eq. (2.8a), to reversing the sense of the turn but retaining the same great circle. To compute the product $u'u$ for arbitrary u' and u, we remark that since any two great circles on S^2 definitely intersect, we can choose $\mathbf{n}, \mathbf{n}', \mathbf{n}'' \in S^2$ such that $u = A(\mathbf{n},\mathbf{n}')$, $u' = A(\mathbf{n}',\mathbf{n}'')$; then by Eq. (2.8b),

$$u'u = A(\mathbf{n}',\mathbf{n}'')A(\mathbf{n},\mathbf{n}') = A(\mathbf{n},\mathbf{n}''). \tag{2.9}$$

Thus the turn for the product is indeed obtained by the geometrical operation described in the Introduction.

We remark that while the geometrical construction is in a three-dimensional Euclidean space (more precisely, on an S^2 therein), we are able to represent SU(2) elements, not merely those of SO(3), faithfully by turns.

To conclude this section, we return to the geometrical meaning of $A(\mathbf{n},\mathbf{n}')$ revealed in Eqs. (2.6) and ask: is there a simple expression for an element $B(\mathbf{n},\mathbf{n}') \in \mathrm{SU}(2)$ such that, unlike with $A(\mathbf{n},\mathbf{n}')$, $R(B(\mathbf{n},\mathbf{n}'))$ will be a rotation about $\mathbf{n}\wedge\mathbf{n}'$ which takes $\mathbf{n}$ precisely to $\mathbf{n}'$ through an angle $\leqslant\pi$? The properties of $A(\mathbf{n},\mathbf{n}')$ tell us that we must take

$$\begin{aligned} B(\mathbf{n},\mathbf{n}') &= A(\mathbf{n},(\mathbf{n}+\mathbf{n}')/|\mathbf{n}+\mathbf{n}'|)\\ &= [2(1+\mathbf{n}\cdot\mathbf{n}')]^{-1/2}(\mathbb{1}+A(\mathbf{n},\mathbf{n}')),\\ A(\mathbf{n},\mathbf{n}') &= (\operatorname{sgn}\mathbf{n}\cdot\mathbf{n}')B(\mathbf{n},\mathbf{n}''), \end{aligned} \tag{2.10}$$

where $\mathbf{n}''$ is as in Eq. (2.6). Then we find

$$\begin{aligned} &B(\mathbf{n},\mathbf{n}')\mathbf{n}\cdot\boldsymbol{\sigma}B(\mathbf{n},\mathbf{n}')^{-1} = \mathbf{n}'\cdot\boldsymbol{\sigma},\\ &B(\mathbf{n},\mathbf{n}')\mathbf{n}\wedge\mathbf{n}'\cdot\boldsymbol{\sigma}B(\mathbf{n},\mathbf{n}')^{-1} = \mathbf{n}\wedge\mathbf{n}'\cdot\boldsymbol{\sigma}. \end{aligned} \tag{2.11}$$

As is to be expected, B is not unambiguously defined when $\mathbf{n}' = -\mathbf{n}$. We may remark that this construction of $B(\mathbf{n},\mathbf{n}') \in \mathrm{SU}(2)$ is useful in computations of the Pancharatnam–Aharonov–Anandan phase[8] for two-level quantum systems, as shown in Ref. 2.

III. GENERALIZATION OF TURNS TO SU(1,1): THEORY OF SCREWS

We now show how a geometrical method can be developed for the noncompact group SU(1,1), similar in spirit to turns for SU(2). For convenience of exposition, this section is divided into subsections.

A. Notational preliminaries and definitions

The defining representation of SU(1,1) consists of two-dimensional complex pseudounitary unimodular matrices of the form

$$\begin{aligned} &A = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix},\\ &\det A = |\alpha|^2 - |\beta|^2 = 1,\\ &A^{\dagger}\sigma_3 A = \sigma_3. \end{aligned} \tag{3.1}$$

The geometrical constructions to follow will be in a three-dimensional Minkowski space $\mathscr{M}$ with vector indices $a,b,... = 0,1,2$ and diagonal metric η_{ab} with signature $(-,+,+)$. By adjoining factors of i to two of the Pauli matrices, we define the matrices ρ_a by

$$\rho_0 = \sigma_3, \quad \rho_1 = i\sigma_1, \quad \rho_2 = i\sigma_2. \tag{3.2}$$

Then

$$\rho_a \rho_b = -\eta_{ab} + i\epsilon_{ab}{}^c \rho_c, \tag{3.3}$$

where ϵ_{abc} is the Levi-Civita symbol with $\epsilon_{012} = 1$. For vectors x, y in $\mathscr{M}$ with inner product $x \cdot y = \eta_{ab} x^a y^b$, define the cross product $z = x \wedge y$ through

$$z^a = \epsilon^a{}_{bc} x^b y^c. \tag{3.4}$$

It is useful to introduce three unit vectors e_a along the coordinate directions in $\mathscr{M}$. For them we have

$$e_a \cdot e_b = \eta_{ab}, \qquad e_a \wedge e_b = \epsilon_{ab}{}^c e_c. \tag{3.5}$$

The following identities are immediate consequences of these definitions:

$$x \cdot y \wedge z = x \wedge y \cdot z = y \cdot z \wedge x; \tag{3.6a}$$

$$x \wedge (y \wedge z) = x \cdot y \; z - x \cdot z \; y; \tag{3.6b}$$

$$x \cdot \rho y \cdot \rho = -x \cdot y + i \; x \wedge y \cdot \rho; \tag{3.6c}$$

$$(x \cdot y)^2 - (x \wedge y)^2 = x^2 y^2; \tag{3.6d}$$

$$w \cdot x \; y \wedge z + w \cdot y \; z \wedge x + w \cdot z \; x \wedge y = (x \cdot y \wedge z) \; w. \tag{3.6e}$$

The last identity arises by expanding $x \wedge (w \wedge (y \wedge z))$ in two different ways.

For a real scalar λ and a real vector μ in $\mathscr{M}$, consider the matrix

$$A = \lambda + i\mu \cdot \rho = \begin{pmatrix} \lambda + i\mu^0 & -\mu^1 + i\mu^2 \\ -\mu^1 - i\mu^2 & \lambda - i\mu^0 \end{pmatrix}. \tag{3.7}$$

It is clear upon comparing with Eq. (3.1) that $A \in \mathrm{SU}(1,1)$ if

$$\det A = \lambda^2 - \mu \cdot \mu = 1, \tag{3.8}$$

and, conversely, any $A \in \mathrm{SU}(1,1)$ can be expressed in the form (3.7) for unique real λ, μ satisfying (3.8). This is the analog of the use of homogeneous Euler parameters for SU(2).

The SU(1,1) to SO(2,1) homomorphism is easily set up. If $B, B' \in \mathrm{SU}(1,1)$, from Eqs. (3.3) and (3.6), it follows that

$$\begin{aligned} &Bx \cdot \rho B^{-1} = x' \cdot \rho, \\ &x'^a = \Lambda(B)^a{}_b x^b, \\ &\Lambda(B) \in \mathrm{SO}(2,1), \\ &\Lambda(B')\Lambda(B) = \Lambda(B'B). \end{aligned} \tag{3.9}$$

Therefore

$$\begin{aligned} &B(\lambda + i\mu \cdot \rho) B^{-1} = \lambda + i\mu' \cdot \rho, \\ &\mu' = \Lambda(B)\mu. \end{aligned} \tag{3.10}$$

Thus under conjugation of $A = \lambda + i\mu \cdot \rho$ by B, the scalar λ is invariant while the vector μ undergoes the SO(2,1) transformation determined by B. The explicit expression for $\Lambda(B)$ in terms of B can be easily worked out, but we do not need it in our analysis.

B. Classification of finite elements of SU(1,1): The exponential map

The representation (3.7) can be used to classify SU(1,1) elements in a convenient form. A vector $\mu \in \mathscr{M}$ is timelike (t), lightlike (l), or spacelike (s) according to whether $\mu^a \mu_a$ is negative, zero, or positive. We will say $A = \lambda + i\mu \cdot \rho \in \mathrm{SU}(1,1)$ is of type t, l, or s according to the nature of μ. This accounts for all elements except $A = \pm \mathbb{1}$ when μ vanishes. In the t and l cases, a further split into positive and negative types depending on the sign of μ^0 is possible. This classification results in five nonintersecting subsets of SU(1,1) whose union along with $\pm \mathbb{1}$ is SU(1,1).

Since $\mu^a \mu_a = \lambda^2 - 1$, the type of an element is fixed by $\lambda = \frac{1}{2} \operatorname{tr} A$. Assuming μ does not vanish identically,

$$\begin{aligned} &-1 < \lambda < 1 &&\Leftrightarrow \quad A \text{ of type } t; \\ &\lambda = \pm 1 &&\Leftrightarrow \quad A \text{ of type } l; \\ &\lambda < -1 \text{ or } \lambda > 1 &&\Leftrightarrow \quad A \text{ of type } s. \end{aligned} \tag{3.11}$$

From Eq. (3.10) it follows that the type of an element is invariant under conjugation by any SU(1,1) element. For the t and l cases, the positive or negative nature is also preserved.

This classification of finite SU(1,1) elements is to be contrasted with a similar classification of elements in the Lie algebra of SU(1,1). The distinction is important because there are finite elements in the group that do not lie on any one-parameter subgroup at all. This is the meaning of the statement that SU(1,1) is not of exponential type. The exponential map takes the Lie algebra of SU(1,1) into a subset of the SU(1,1) group manifold. The complement of the range of this map can be easily characterized. The one-parameter subgroups of the types t, l, and s (with parameter τ) are

$$\begin{aligned} &A_t(u;\tau) = \cos\tau + iu \cdot \rho \sin\tau, \quad 0 \leqslant \tau \leqslant 2\pi, \quad u^2 = -1; \\ &A_l(u,\tau) = 1 + iu \cdot \rho\tau, \quad \tau \in \mathbb{R}, \quad u^2 = 0; \\ &A_s(u;\tau) = \cosh\tau + iu \cdot \rho \sinh\tau, \quad \tau \in \mathbb{R}, \quad u^2 = 1. \end{aligned} \tag{3.12}$$

Upon comparison with the classification of SU(1,1) elements, we see that the range of the exponential map consists of all elements of type t, elements of type l with $\lambda = 1$, of type s with $\lambda > 1$, and the two elements $\pm \mathbb{1}$. The complement of this range is therefore

$$\Phi = \{l \text{ type with } \lambda = -1\} \cup \{s \text{ type with } \lambda < -1\}. \tag{3.13}$$

In short, $\lambda = \frac{1}{2} \operatorname{tr} A \leqslant -1$ and $\mu \neq 0$ implies that A does not lie on any one-parameter subgroup; nevertheless A itself can be classified. We may also note that if $A \in \Phi$, then $A^{-1} \in \Phi$ as well.

In the sequel we will see the importance of this classification of SU(1,1) elements for the theory of screws.

C. Definition of the screw

Guided by the definition of turns for SU(2), for any two real three-vectors $x, y \in \mathscr{M}$ we define a complex two-dimensional matrix

$$A(x,y) = x \cdot y + i \; x \wedge y \cdot \rho. \tag{3.14}$$

From the identity (3.6d) we see that

$$\det A(x,y) = x^2 y^2. \tag{3.15}$$

Since $A(x,y)$ is of the general form (3.7), we see that it is an element of SU(1,1) if and only if $x^2 y^2 = 1$. That is, both x and y must be spacelike or both timelike, and have reciprocal norms. Henceforth we restrict our attention to spacelike vectors; the reason for this will be clear when we study the composition of screws. We next use the freedom of reciprocal scalings of x and y to arrange them both to be unit spacelike vectors.

In $\mathscr{M}$ define the single-sheeted spacelike unit hyperboloid

$$\Sigma = \{x | x\cdot x \equiv (x^1)^2 + (x^2)^2 - (x^0)^2 = 1\}. \tag{3.16}$$

The role of S^2 on which turns were defined for SU(2) will now be played by Σ. We know from Eqs. (3.14) and (3.15) that

$$x,y\in\Sigma \Rightarrow A(x,y)\in \mathrm{SU}(1,1). \tag{3.17}$$

We wish to prove the converse: any $A\in\mathrm{SU}(1,1)$ is of the form $A(x,y)$ for suitable x,y on Σ:

$$A \equiv \lambda + i\mu\cdot\rho = A(x,y) \equiv x\cdot y + ix\wedge y\cdot\rho. \tag{3.18}$$

If this is to be so, evidently, both x and y must be Lorentz orthogonal to μ, and also lie on Σ. Therefore for each μ we define

$$\mathscr{C}(\mu) = \{x | x\cdot x = 1, \quad \mu\cdot x = 0\}. \tag{3.19}$$

Clearly $\mathscr{C}(\mu)$ is the intersection of Σ with the plane through the origin orthogonal to μ. For μ of type t, l, or s, $\mathscr{C}(\mu)$ consists, respectively, of a closed ellipse, a pair of parallel disjoint infinite straight lines (generators of Σ), or a pair of disjoint infinite hyperbolas.

Theorem: Given any $A = \lambda + i\mu\cdot\rho\in\mathrm{SU}(1,1)$, we can choose $x, y\in\mathscr{C}(\mu)$ such that $A(x,y) = \lambda + i\mu\cdot\rho$. Moreover, either x or y can be chosen arbitrarily on $\mathscr{C}(\mu)$, the other being then uniquely fixed.

Proof: Since our entire treatment is manifestly SO(2,1) covariant, and under conjugation we have the behavior given in Eq. (3.10), we can with no loss of generality put μ of each type into a convenient standard configuration, and then carry out the construction.

<u>A of type t:</u> We can take

$$\mu = \xi e_0, \quad \xi = \sin\theta, \quad \lambda = \cos\theta, \quad 0<\theta<2\pi, \quad \theta\neq\pi. \tag{3.20}$$

Then

$$\mathscr{C}(\mu) = \{x | x^0 = 0, \quad (x^1)^2 + (x^2)^2 = 1\}$$
$$= \{(0,\cos\varphi, \quad \sin\varphi) | 0\leqslant\varphi<2\pi\}. \tag{3.21}$$

If $x, y\in\mathscr{C}(\mu)$ correspond to parameter values φ,φ', respectively,

$$\lambda + i\mu\cdot\rho = A(x,y) \Leftrightarrow$$
$$\lambda = x\cdot y, \quad \mu = x\wedge y \Leftrightarrow \tag{3.22}$$
$$\varphi - \varphi' = \theta.$$

Clearly, if θ is known, either x or y can be freely chosen on $\mathscr{C}(\mu)$, the other is then uniquely fixed.

<u>A of type l:</u> We can set

$$\mu = \epsilon'(e_0 + e_2), \quad \lambda = \epsilon, \quad \epsilon,\epsilon' = \pm 1. \tag{3.23}$$

Then $\mathscr{C}(\mu)$ is a pair of generators of Σ:

$$\mathscr{C}(\mu) = \{x | x^0 = x^2, \quad (x^1)^2 = 1\}$$
$$= \{(a,\delta,a) | a\in\mathbb{R}, \quad \delta = \pm 1\}. \tag{3.24}$$

By taking $x = (a,\delta,a)$ and $y = (a',\delta',a')$,

$$\lambda + i\mu\cdot\rho = A(x,y) \Leftrightarrow,$$
$$\delta\delta' = \epsilon, \quad a\delta' - a'\delta = \epsilon'. \tag{3.25}$$

If ϵ and ϵ' are known, any choice of (a,δ) leads to unique (a',δ') and the converse. If $\epsilon = 1$, x and y are on the same branch of $\mathscr{C}(\mu)$; if $\epsilon = -1$, they are not.

<u>A of type s:</u> We can take

$$\mu = \xi e_2, \quad \xi = \epsilon\sinh\zeta, \quad \lambda = \epsilon\cosh\zeta,$$
$$\epsilon = \pm 1, \quad \zeta\in\mathbb{R}, \quad \zeta\neq 0. \tag{3.26}$$

Then $\mathscr{C}(\mu)$ consists of two branches of a hyperbola on Σ:

$$\mathscr{C}(\mu) = \{x | x^2 = 0, \quad (x^1)^2 - (x^0)^2 = 1\}$$
$$= \{(\delta\sinh\eta, \quad \delta\cosh\eta, 0) | \eta\in\mathbb{R},$$
$$\delta = \pm 1\}. \tag{3.27}$$

By choosing parameters δ, η for x and δ', η' for y,

$$\lambda + i\mu\cdot\rho = A(x,y) \Leftrightarrow \delta\delta' = \epsilon, \quad \eta - \eta' = \zeta. \tag{3.28}$$

Once again, either x or y can be freely chosen on $\mathscr{C}(\mu)$, the other being then completely determined. If $\epsilon = +1$, x and y lie on the same branch of $\mathscr{C}(\mu)$, otherwise not.

This completes the proof, except for the remark that for $A = \pm\mathbb{1}$, we can set $x = \pm y$ on Σ and then choose x freely, i.e.,

$$x\in\Sigma : A(x, \pm x) = \pm\mathbb{1}. \tag{3.29}$$

We are now able to define a screw precisely. Notice that with $A = \lambda + i\mu\cdot\rho\in\mathrm{SU}(1,1)$, the SO(2,1) "rotation" $\Lambda(A)$ leaves $\mu\in\mathscr{M}$ invariant, and alters only vectors in the plane orthogonal to μ. By borrowing the familiar SO(3) language, we can say that $\Lambda(A)$ is a Lorentz rotation about μ as axis. A screw is then an equivalence class of ordered pairs of points (x,y) on Σ, the equivalence being with respect to common SO(2,1) transformations of both x and y about $x\wedge y$ as axis and, in case $x\wedge y$ is not of type t, the transformation $x\to -x$, $y\to -y$ as well. Since $A(x,y)$ is clearly invariant under such transformations, it follows from the above theorem that there is a one-to-one correspondence between screws and SU(1,1) elements, determined by Eq. (3.14). Given $x,y\in\Sigma$ with $x\neq\pm y$, there is a unique $\mathscr{C}(\mu)$ on which x and y lie, and then the equivalence is with respect to motions along $\mathscr{C}(\mu)$ induced by SO(2,1) rotations about $\mu = x\wedge y$, along with the "reflection" $x\to -x$, $y\to -y$ in case μ is not of type t. For $A = \mathbb{1}$, we have the "null screw" given by (x,x) for any $x\in\Sigma$; and for $A = -\mathbb{1}$, any pair $(x, -x)$ can be used.

D. Screws as directed arcs on Σ

We have just defined a screw as an equivalence class of ordered pairs of points (x,y) on Σ. One may wish, analogous to turns for SU(2), to define a screw as (the equivalence class of) the directed arc from x to y along $\mathscr{C}(\mu = x\wedge y)$. However, in the proof of the theorem of the previous subsection, we found that among l- and s-type elements of SU(1,1), there are situations when x and y have to be chosen on distinct branches of $\mathscr{C}(\mu)$. Now we see that these elements are precisely the ones in $\Phi\subset\mathrm{SU}(1,1)$, the complement of the range of the exponential map. So at first sight it appears that for elements in Φ, screws cannot be visualized as connected arcs on Σ. There is, however, a way of overcoming this problem, which does not exist for turns for SU(2). The clue is that A and $-A$ in SU(1,1) share the same $\mathscr{C}(\mu)$, and if $A \in \Phi$, then $-A \notin \Phi$. Hence, if $A \in \Phi$, the ordered pair of points corresponding to $-A$ can definitely be connected by a directed arc along $\mathscr{C}(\mu)$. This leads to an alternative definition of a screw.

A screw is a pair consisting of an equivalence class of connected directed arcs along a $\mathscr{C}(\mu)$, the equivalence being with respect to SO(2,1) transformations that map $\mathscr{C}(\mu)$ onto itself and with respect to reflection if μ is not of type t; and a flag that can take the values ± 1. Given $A = A(x,y) \notin \Phi$, its screw is (the equivalence class of) the directed arc from x to y along $\mathscr{C}(x \wedge y)$, with the flag $+1$; if however $A = A(x,y) \in \Phi$, then the screw is represented by (the class of) the directed arc for $-A = A(x, -y) \notin \Phi$, with the flag -1.

We shall return to this use of the flag after discussing the geometrical composition procedure for screws.

E. Composition of screws

From the representation (3.7) for a general $A \in$ SU(1,1), and the properties of the matrices ρ, we can see that passage to the inverse corresponds to reversing the sign of μ but leaving λ unchanged:

$$A = \lambda + i\mu\cdot\rho \Rightarrow A^{-1} = \lambda - i\mu\cdot\rho. \tag{3.30}$$

It is then clear that, given the screw for A, the screw for A^{-1} is obtained by interchanging the entries in the ordered pair of vectors (x,y), x and $y \in \mathscr{C}(\mu)$, or equivalently by reversing the directed arc without altering the flag.

To develop a geometrical rule for the composition of screws, we first give an algebraic result following from the construction (3.14), the form of which is suggested by the result (2.8b) in the SU(2) case:

$$x,y,z \in \mathscr{M}: \quad A(z,y)A(x,z) = z^2 A(x,y). \tag{3.31}$$

The proof is quite straightforward, and involves judicious use of the various identities (3.6). In fact, the precise construction of $A(x,y)$ in Eq. (3.14) was motivated by the desire to have the result (3.31). The idea now is to see if (3.31) can be exploited to convert the rule for composition of any two SU(1,1) elements into a geometrical operation on Σ.

If two elements $A, A' \in$ SU(1,1) are expressed in the form (3.7), their product can be put into the same form:

$$\begin{aligned} A'' &= A'A = (\lambda' + i\mu'\cdot\rho)(\lambda + i\mu\cdot\rho) = \lambda'' + i\mu''\cdot\rho, \\ \lambda'' &= \lambda'\lambda + \mu'\cdot\mu, \\ \mu'' &= \lambda'\mu + \lambda\mu' - \mu'\wedge\mu. \end{aligned} \tag{3.32}$$

On the other hand, if in Eq. (3.31) all three vectors x,y,z are chosen on Σ, then both $A(x,z)$ and $A(z,y)$ will belong to SU(1,1) and

$$x,y,z \in \Sigma: A(z,y)A(x,z) = A(x,y). \tag{3.33}$$

We can conclude that if, given the two general elements A, $A' \in$ SU(1,1), we determine $\mathscr{C}(\mu)$ and $\mathscr{C}(\mu')$ and find that they intersect, we can then choose $z \in \mathscr{C}(\mu) \cap \mathscr{C}(\mu')$; from Sec. III C we are then assured that $x \in \mathscr{C}(\mu)$ and $y \in \mathscr{C}(\mu')$ exist uniquely such that $A = (x,z)$ and $A' = A(z,y)$. Then $A'A$ is determined by Eq. (3.33): the ordered pair for $A'A$ consists of the first member of the A pair and the second member of the A' pair, in that order. However, unlike the SU(2) case where we know that any two great circles on S^2 definitely intersect, in the Lorentzian geometry of $\mathscr{M}$ it does sometimes happen that $\mathscr{C}(\mu) \cap \mathscr{C}(\mu') = \phi$! We are thus led to the question: when is $\mathscr{C}(\mu) \cap \mathscr{C}(\mu') \neq \phi$?

In general, if two vectors $\mu, \mu' \in \mathscr{M}$ are given, assumed to be linearly independent, then the planes through the origin orthogonal, respectively, to μ and to μ' will intersect along a straight line:

$$\mu\cdot x = \mu'\cdot x = 0 \Rightarrow x = \alpha\mu\wedge\mu'. \tag{3.34}$$

Here α is a parameter along the line. This line will cut Σ if there is a real value of α for which

$$\alpha^2(\mu\wedge\mu')^2 \equiv \alpha^2((\mu\cdot\mu')^2 - \mu^2\mu'^2) = 1, \tag{3.35}$$

which will happen if and only if

$$(\mu\cdot\mu')^2 - \mu^2\mu'^2 > 0. \tag{3.36}$$

We can now systematically analyze the six possible kinematical situations, listing the nature of the pair μ,μ' at tt, tl, ts, ll, ls, and ss, and check in each whether the inequality (3.36) can be obeyed. (Naturally, in the "off-diagonal" cases, such as tl, it does not matter which of μ and μ' is of type t and which of type l.) Keeping in mind the nature of Lorentzian geometry, we find that in four situations, the inequality is uniformly obeyed:

$$tt,tl,ts,ll: (\mu'\cdot\mu)^2 - \mu'^2\mu^2 > 0. \tag{3.37}$$

However, in the two remaining cases ls and ss, no uniform statement can be made; depending on the specific choices of μ and μ', $(\mu'\cdot\mu)^2 - \mu'^2\mu^2$ could have either sign.

The result is that if for the two elements A, A' in SU(1,1), at least one of the vectors μ, μ' is of type t, or if both are of type l, $\mathscr{C}(\mu')$ and $\mathscr{C}(\mu)$ definitely intersect on Σ; then Eq. (3.33) is adequate to give us a geometrical "addition" or composition law for screws, in a way similar to the use of Eq. (2.8b) for SU(2). But in the ls and ss cases, there is no guarantee that $\mathscr{C}(\mu')$ and $\mathscr{C}(\mu)$ will intersect; and if they do not, a choice of one common vector z as in Eq. (3.33) is not possible.

Fortunately the following decomposition theorem, which exhibits an interesting structural property of SU(1,1) comes to our aid, so that Eq. (3.33) can be used for composing screws in all situations:

Theorem: Any $A'' \in$ SU(1,1) can be expressed (in infinitely many ways) as a product $A'' = A'A$ where both factors A' and A are of type t.

Proof: We naturally construct the argument in the spirit of the theory of screws, and use the geometrical machinery in $\mathscr{M}$ already developed. Assuming $A'' \neq \pm 1$, we identify $\mu'' \neq 0$, construct $\mathscr{C}(\mu'') \subset \Sigma$, and pick $x,y \in \mathscr{C}(\mu'')$ such that

$$A'' = \lambda'' + i\mu''\cdot\rho = A(x,y). \tag{3.38}$$

Here x and y will be linearly independent, $x \neq \pm y$. Since each of them is spacelike, there are infinitely many timelike vectors n orthogonal to x, and similarly n' orthogonal to y. There are then infinitely many choices of n and n' obeying

$$\begin{aligned} &n^2, n'^2 < 0; \quad n\cdot x = n'\cdot y = 0; \\ &n\cdot y \neq 0, \quad n'\cdot x \neq 0. \end{aligned} \tag{3.39}$$

These conditions are designed to ensure that n and n' are linearly independent and neither is proportional to $x\wedge y$. Since both n and n' are timelike, $n\wedge n'$ is spacelike and so when normalized will cut Σ:

$$z = \alpha n\wedge n' \in \Sigma, \quad \alpha = ((n\cdot n')^2 - n^2 n'^2)^{-1/2}. \tag{3.40}$$

Now we construct the SU(1,1) elements $A = A(x,z)$, $A' = A(z,y)$. Both of them are of type t since by Eq. (3.39)

$$x\wedge z = \alpha x\wedge(n\wedge n') = -\alpha x\cdot n' n, \quad z\wedge y = \alpha(n\wedge n')\wedge y = -\alpha y\cdot n n', \tag{3.41}$$

are both nonvanishing timelike vectors. Finally,

$$A'' = A(x,y) = A(z,y)A(x,z) = A'A, \tag{3.42}$$

which proves the result for $A''\neq \pm 1$. If $A'' = \pm 1$, we can take A to be any element of type t, and $A' = \pm A^{-1}$.

To see by way of illustration how the choices of A and A' may be made, we can consider in turn A'' to be of type t,l,s. If A'' is already of type t, it lies on a t-type, one-parameter subgroup, so for instance A and A' could be chosen equal and essentially the square root of A''. If A'' is of type l or s, the situation is nontrivial. In these cases we can use the manifest covariance under SO(2,1), i.e., the conjugation relation (3.10), and assume without loss of generality that λ'' and μ'' are in some standard configuration. This makes possible choices of x,y,n,n',z easy to visualize. We record below the standard forms of λ'' and μ'', possibilities for n and n', and the resulting z, leaving it to the reader to check that all conditions (3.32), (3.38)–(3.40), and (3.42) are obeyed.

A'' of type l:

$$\begin{aligned} &\lambda'' = \epsilon, \quad \mu'' = \epsilon'(e_0 + e_2), \quad \epsilon,\epsilon' = \pm 1;\\ &x = e_1, \quad y = \epsilon e_1 - \epsilon'(e_0+e_2);\\ &n = e_0, \quad n' = \epsilon'(2e_0 + e_2) - \epsilon e_1;\\ &z = (\epsilon' e_1 + \epsilon e_2)/\sqrt{2};\\ &\lambda = \epsilon'/\sqrt{2}, \quad \mu = -\epsilon e_0/\sqrt{2};\\ &\lambda' = 0, \quad \mu' = (2e_0 - \epsilon\epsilon' e_1 + e_2)/\sqrt{2}. \end{aligned} \tag{3.43}$$

A'' of type s:

$$\begin{aligned} &\lambda'' = \epsilon\cosh\zeta, \quad \mu'' = \epsilon\sinh\zeta e_2, \quad \epsilon = \pm 1, \quad \zeta\neq 0;\\ &x = e_1, \quad y = \epsilon(-\sinh\zeta e_0 + \cosh\zeta e_1);\\ &n = e_0, \quad n' = \cosh\zeta e_0 - \sinh\zeta e_1;\\ &z = e_2;\\ &\lambda = 0, \quad \mu = -e_0;\\ &\lambda' = 0, \quad \mu' = \epsilon(\cosh\zeta e_0 - \sinh\zeta e_1). \end{aligned} \tag{3.44}$$

With the help of our theorem, then, the product AB of any two elements $A,B\in$ SU(1,1) can be handled geometrically as an operation on screws, requiring at most two applications of Eq. (3.33). If A, B belong to one of the four cases tt, tl, ts, or ll, we "slide" the representative pairs of points on $\mathscr{C}(\mu_A)$, $\mathscr{C}(\mu_B)$ till the "head" (second element) of the B pair and the "tail" (first element) of the A pair become $z\in\mathscr{C}(\mu_A)\cap\mathscr{C}(\mu_B)$. Then a single use of Eq. (3.33) gives the screw for AB as determined by the pair (tail of B, head of A). If A,B belong to either the ls or ss cases, and $\mathscr{C}(\mu_A)\cap\mathscr{C}(\mu_B) = \phi$, we use the decomposition theorem to write $AB = A''A'B$, with both A'' and A' being of type t. The screws for A' and B can then be composed using (3.33) to give the screw for $A'B$; this can then be composed with the screw for A'', using (3.33) again, to give the final result.

We now go back to the device of the flag, introduced in Sec. III D so as to allow us to visualize *every* $A\in$ SU(1,1) as a connected arc on Σ plus a flag, and show how it can be represented graphically. As noted in Eq. (3.29), the element $-\mathbb{1}$ in SU(1,1) corresponds to the (degenerate) screw determined by any pair $(x,-x)$:

$$x\in\Sigma: \quad A(x,-x) = -\mathbb{1}. \tag{3.45}$$

This is just like the great semicircle turn in the SU(2) case. Since x is spacelike, there are infinitely many timelike vectors μ orthogonal to it, which means there are infinitely many connected $\mathscr{C}(\mu)$'s of type t containing both x and $-x$. [In addition, there are infinitely many s-type $\mathscr{C}(\mu)$'s, and two l-type $\mathscr{C}(\mu)$'s, each made up of two branches, and each containing x and $-x$, but on separate branches; these however are not useful for the present purpose.] This degenerate screw, representable by a connected arc on any one of these t-type $\mathscr{C}(\mu)$'s, and in fact running halfway across it, is indeed the flag we have used above.

As an example, consider the element

$$A = \begin{pmatrix} -\cosh\theta & -\sinh\theta \\ -\sinh\theta & -\cosh\theta \end{pmatrix} \in \Phi \subset \mathrm{SU}(1,1), \tag{3.46}$$

for which

$$\lambda = -\cosh\theta, \quad \mu = e_1\sinh\theta. \tag{3.47}$$

Then $\mathscr{C}(\mu)$ consists of two branches of the hyperbola

$$(x^2)^2 - (x^0)^2 = 1 \tag{3.48}$$

in the 0-2 plane, as in Fig. 1. A choice of x,y so that $A = A(x,y)$ is

$$x = (\sinh(\theta-\theta_0), \quad 0, \quad \cosh(\theta-\theta_0)), \quad y = (\sinh\theta_0, \quad 0, \quad -\cosh\theta_0), \tag{3.49}$$

where θ_0 may be chosen freely. Naturally they are on sepa-

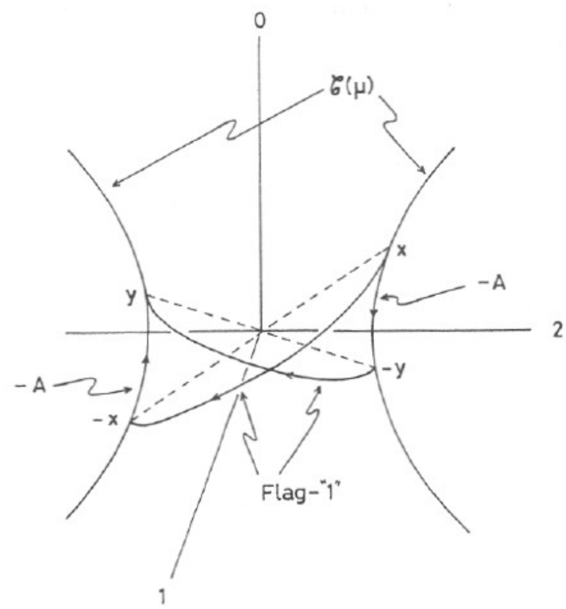

FIG. 1. Use of the flag $-$ "1."

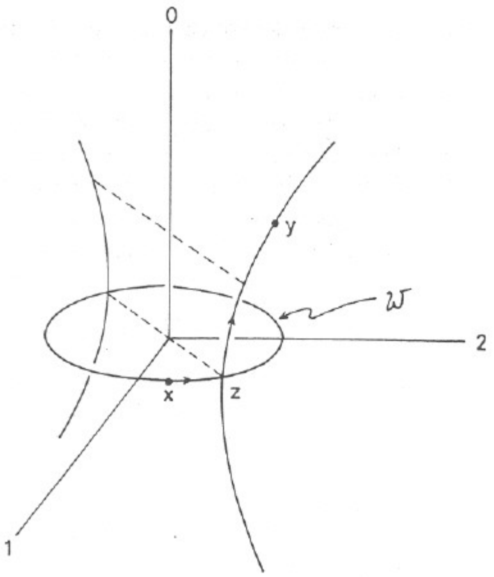

FIG. 2. Rotation boost decomposition of general element.

rate branches of $\mathscr{C}(\mu)$. Now, $-A = A(x,-y) = A(-x,y)$ is in the range of the exponential map, and can be represented by the connected arc $x \rightarrow -y$ or $-x \rightarrow y$ as we wish, along one branch of $\mathscr{C}(\mu)$. Coupled, respectively, with the flag screws $A(-y,y)$ or $A(x,-x)$ standing for $-\mathbb{1}$, we get back $A(x,y)$ via Eq. (3.33):

$$A = A(x,y) = A(-y,y)A(x,-y) = A(-x,y)A(x,-x). \qquad (3.50)$$

This is graphically seen in Fig. 1.

IV. CONCLUDING REMARKS

In this paper we have presented a generalization of Hamilton's method of turns for SU(2) to a theory of screws for SU(1,1), thus leading to a new and useful way of picturing the elements and structure of this noncompact group. The two distinguishing features of SU(1,1), as contrasted with SU(2), are that the range of the exponential map is a subset of SU(1,1); and that two planes passing through the origin in $\mathscr{M}$ may in some cases have a line of intersection that does not cut Σ. These features, which at first sight seem to pose problems for the development of a complete geometrical picture, can be taken care of by the use of the flag -1 when appropriate, and the use of the theorem of Sec. III E, so that all group elements and their products can be satisfactorily handled.

As an example of the usefulness of our geometrical picture for SU(1,1), we recall the result that any element of SU(1,1) is a "boost" in an appropriate direction followed by a rotation [element of the maximal compact subgroup U(1)], or a rotation followed by a boost. This fact is immediately and visually obvious in the screw representation, requiring no calculation at all! Elements of the U(1) subgroup have μ parallel to e_0, so for them $\mathscr{C}(\mu)$ is the unit circle in the 1-2 plane, or the "waist" $\mathscr{W}$ of Σ:

$$\mathscr{W}: \quad x^0 = 0, \quad (x^1)^2 + (x^2)^2 = 1.$$

This is like the equator on S^2 in the turns case. On the other hand, pure boosts are elements for which μ is a linear combination of e_1 and e_2. For such a μ, $\mathscr{C}(\mu)$ is the intersection of Σ with a "vertical" plane containing the e_0 axis; the two branches of the hyperbola making up $\mathscr{C}(\mu)$ are like "lines of longitude" on S^2. Now given a screw determined by the pair (x,y), we can assume without loss of generality that either x or y, whichever we wish, lies on $\mathscr{W}$. This is because *every* $\mathscr{C}(\mu)$ is guaranteed to intersect $\mathscr{W}$. Assuming for definiteness then that $x \in \mathscr{W}$, we can draw a hyperbola on Σ in the vertical plane containing y and the e_0 axis. If this cuts $\mathscr{W}$ at a point z, we can recover the pair (x,y) by composing the screws (x,z) and (z,y) in that order. [Note incidentally that both these are connected arcs, whereas no assumption was made about x and y being on a connected branch of $\mathscr{C}(x,y)$.] This is the proof by the present method that any SU(1,1) element is a pure rotation followed by some pure boost, as depicted in Fig. 2. If we had $y \in \mathscr{W}$ instead, the decomposition would have been in the opposite order.

We may note that the t,l,s classification of finite SU(1,1) elements is relevant to periodically focusing optical systems.[9] An example is a laser resonator. One sees that the system is stable if and only if the ray-transfer matrix for one period is a t-type element of SL$(2,R)$. An interesting application of the theorem of subsection 3.5 is to squeezing that is an s-type element of SU(1,1). The theorem shows that squeezing can be realized by switching periodically between two t-type nonsqueezing transformations. A detailed analysis of these questions, as well as the development of a new representation for first-order Fourier optical systems, will be presented elsewhere.

ACKNOWLEDGMENT

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