Complex static solutions and wave solutions to (2+1)-dimensional topologically massive gauge theories

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Complex static solutions with zero action and zero energy–momentum tensor are constructed for the (2+1)-dimensional topologically massive SU(2) gauge theory by generalizing the vortex-like solutions. These solutions can be treated as real gauge potentials for the noncompact gauge group SL(2, ℂ). Wave solutions are also exhibited.

Recently there has been a surge of interest in (2+1)-dimensional field theories defined by the Chern–Simons (CS) action [1]. The CS action is a covariant action and a useful tool to study two-dimensional conformal field theories, integrable lattice models, fractional statistics and knot theory in a three-dimensional setting. Furthermore in (2+1)-dimensional gravity, the Einstein–Hilbert action is the CS action with the appropriate gauge groups. Thus the CS theory with noncompact gauge group SL(2, ℂ) can be treated as (2+1)-dimensional general relativity with positive cosmological constant and Lorentz signature [2]. At the classical level, the CS theory has a trivial solution; in this note we shall search for classical solutions to theories defined by the CS action together with the Yang–Mills (YM) action, i.e. topologically massive gauge field theories [3]. In passing we merely note that the CS theory can be regarded as an effective low energy limit of the topologically massive gauge theory or one may view the YM action as a higher derivative term to regulate the perturbative CS theory [4].

In ref. [5], vortex-like solutions for the SU(2) YM equations with the CS term are obtained. Here we shall first exhibit a general ansatz for vortex-like solutions; this ansatz is then suitably generalized so as to give complex solutions such that the vortex-like behaviour at large distances is retained. Now according to ref. [6], complex gauge fields A for the gauge groups SU(2) can be understood as the real gauge fields b for the noncompact group SL(2, ℂ) by taking the real and imaginary parts of A as the first three and second three components of a real gauge field b. Furthermore the total lagrangian $\mathscr{L}$ for the gauge groups SL(2, ℂ) is given by the real part of $(e^{i\beta} L)$, where L is the complex SU(2) lagrangian and β is a real parameter. In this way we arrive at the real solutions for the SL(2, ℂ) YM equations with CS term. The solutions give rise to zero action and zero energy–momentum tensor; however, the magnetic flux and the total nonabelian electric charge can be nonvanishing. In general the energy for a SL(2, ℂ) YM theory is not positive definite; however, since a suitable low-energy limit of the topologically massive YM theory yields the CS theory we then need not worry about the positive energy requirement as the hamiltonian in the CS theory is zero [2]. In the last part of this note we shall discuss some wave-like solutions. We end with some remarks.

The action for a SU(2) topologically massive YM field theory is given by

$$S=\int d^3x(L_{\mathrm{YM}}+L_{\mathrm{CS}}), \tag{1a}$$

$$L_{\mathrm{YM}}=-\tfrac{1}{4}F^{\mu\nu}_{a}F^{a}_{\mu\nu}, \tag{1b}$$

$$L_{\mathrm{CS}}=\tfrac{1}{2}\xi\epsilon^{\mu\nu\alpha}(\partial_\mu A^a_\nu A^a_\alpha+\tfrac{1}{3}\epsilon^{abc}A^a_\mu A^b_\nu A^c_\alpha), \tag{1c}$$

where for convenience we set the gauge field coupling

constant $g=1$ and the metric is $g_{\mu\nu}=(-++)$. The equations of motion are

$$D_\mu F^{\mu\nu}=j^\nu, \tag{2a}$$

$$j^\nu=-\xi\tilde{F}^\nu\equiv-\tfrac{1}{2}\xi\epsilon^{\nu\alpha\beta}F_{\alpha\beta}, \tag{2b}$$

$$D_\nu j^\nu=D_\nu\tilde{F}^\nu=0. \tag{2c}$$

In euclidean spacetime, ξ is replaced by $-\mathrm{i}\xi$. For vortex-like solutions, the gauge potentials at large distance are characterized by

$$A^i(x)\approx\phi^i/\rho,\quad i=1,2, \tag{3}$$

where $\rho=(x_1^2+x_2^2)^{1/2}$ and ϕ^i denotes the unit vector $\epsilon^{ij}x^j/\rho\equiv\epsilon^{ij}n^j$. In ref. [5] the vortex-like solution is given by

$$A_j^a=(\phi^a C+\delta_3^a/\rho)\phi_j, \tag{4a}$$

$$A_0^a=\phi^a H, \tag{4b}$$

when C and H are functions of ρ and are respectively expressed in terms of the modified Bessel functions of the third kind $K_1(z)$ and $K_0(z)$ with $z\equiv\xi\rho$. A more general ansatz than (4) which also gives rise to vortex-like solutions can be written as

$$A_j^a=\omega^a(C\phi_j+Dn_j)+(\delta_3^a/\rho)\phi_j, \tag{5a}$$

$$A_0^a=\omega^a H, \tag{5b}$$

where $\omega^a=(n^a+\phi^a)$ with $n^3=\phi^3=0$. The function D depends on ρ only and can be gauge-transformed away since it does not appear in the reduced equations of motion. Expressions (5) yield the field strengths

$$E_i^a=F_{i0}^a=\omega^a n_i H', \tag{6a}$$

$$B^a=\tfrac{1}{2}\epsilon_{ij}F^{aij}=-\omega^a(C'+C/\rho)=\omega^a\xi H, \tag{6b}$$

where H' indicates $\mathrm{d}H/\mathrm{d}\rho$ and in the last step of eq. (6b) we have made use of the equation of motion. As before one finds that C and H are respectively the modified Bessel functions $K_1(z)$ and $K_0(z)$.

We proceed to further generalize the ansatz (4) by writing

$$A_j^a=\theta^a(C\phi_j+Dn_j)+\delta_3^a(d/\rho)\phi_j, \tag{7a}$$

$$A_0^a=\theta^a H, \tag{7b}$$

$$\theta^a=(n^a+\mathrm{i}\phi^a), \tag{7c}$$

where d is a constant. Substituting expressions (7) into eqs. (2), the reduced equations are

$$H''+H'/\rho-\left(\frac{d-1}{\rho}\right)^2 H+\xi R=0, \tag{8a}$$

$$R+\xi H=0, \tag{8b}$$

$$R\equiv C'+C/\rho+\mathrm{i}(1-d)D/\rho. \tag{8c}$$

For $d\neq1$ and $\nu=d-1$, we find

$$H=K_\nu(z),\quad D=\mathrm{i}C=\mathrm{i}K_{\nu+1}(z) \tag{9a}$$

or

$$H=K_\nu(z),\quad D=-\mathrm{i}C=-\mathrm{i}K_{\nu-1}(z). \tag{9b}$$

Special cases are

(a) $d=1,\quad H=K_0(z),\quad C=K_1(z),$

$D=$arbitrary function of ρ, (10a)

(b) $D=0,\quad H=K_0(z),\quad C=K_1(z),$ (10b)

(c) $C=0\ (d\neq1),\quad H=K_\nu(z),$

$D=\mathrm{i}zK_\nu(z)/(1-d),\quad C=0\ (d=1),$

$H=K_\nu(z),\quad D=$arbitrary function of ρ. (10c)

The electric and magnetic field strengths are

$$E_i^a=\theta^a[n_i H'+\mathrm{i}\phi_i H(d-1)/\rho], \tag{11a}$$

$$B^a=\theta^a\xi H. \tag{11b}$$

From the behaviour of the modified Bessel function [7], we find that as $z\to0$, E_i^a and B^a are singular ($E_i^a\approx z^{-\nu-1}$, $B^a\approx z^{-\nu}$) whilst at large distances they vanish exponentially fast. Note that solutions (7) and (9) still retain the vortex-like behaviour at large distances. It is easy to verify that the total action vanishes and the energy–momentum tensor

$$\theta_{\mu\nu}=F_{\mu\beta}^a F_\nu^{a\beta}+g_{\mu\nu}L \tag{12}$$

is also zero. Consequently solution (7) carries no angular momentum. However, the total nonabelian electric charge and magnetic flux need not be zero. Projecting along the unit vector n^a, the gauge invariant total electric charge is given by

$$Q=\int \mathrm{d}^2x(\partial_i E^{ai}\cdot n_a)$$

$$=2\pi\int_0^\infty z\,\mathrm{d}z\left(1+\frac{\nu(\nu+1)}{z^2}\right)K_\nu(z),$$

and the magnetic flux is

$$\Phi=\int \mathrm{d}^2x\, B^a n^a=\frac{2\pi}{\xi}\int_0^\infty z\,\mathrm{d}z\, K_\nu(z),$$

which, for general ν, are divergent because of the singularity at $z=0$. Note that for $\nu=0$ $(d=1)$, Q and Φ are proportional to each other, $Q=\xi\Phi=2\pi$.

The ansatz (7) is also valid in euclidean spacetime with A_0^a being replaced by A_3^a. The solutions for $d\neq 1$ are

$$H=Z_\nu,\quad D=\mathrm{i}C=-Z_{\nu+1}(z) \tag{13a}$$

or

$$H=Z_\nu(z),\quad D=-\mathrm{i}C=-Z_{\nu-1}(z), \tag{13b}$$

where $Z_\nu(z)$ denotes the Bessel functions $J_\nu(z)$ or $N_\nu(z)$.

As explained in ref. [6], the real gauge field potentials b_μ^a for the group SL(2, ℂ) can be derived from the complex potentials (7). Thus

$$b_j^a=(n^a\phi_j-\phi^a n_j)C+\delta_3^a(d/\rho)\phi_j, \tag{14a}$$

$$b_j^{a+3}=\delta_j^a C, \tag{14b}$$

$$b_0^a=n^a H, \tag{14c}$$

$$b_0^{a+3}=\phi^a H \tag{14d}$$

where we have made use of eq. (9a). As demonstrated in ref. [6], the YM lagrangian $\mathscr{L}_{\mathrm{YM}}$ for SL(2, ℂ) is the real part of $(\mathrm{e}^{-\mathrm{i}\beta}L_{\mathrm{YM}})$. Similarly one can show that $\mathscr{L}_{\mathrm{CS}}$ for SL(2, ℂ) is also the same as the real part of $(\mathrm{e}^{\mathrm{i}\beta}L_{\mathrm{CS}})$. Indeed one finds for SL(2, ℂ)

$$\begin{aligned}\mathscr{L}_{\mathrm{CS}}=\tfrac{1}{2}\xi\epsilon^{\mu\nu\alpha}\{&\cos\beta[\partial_\mu U_\nu^a U_\alpha^a-\partial_\mu V_\nu^a V_\alpha^a\\&+\epsilon^{abc}(\tfrac{1}{3}U_\mu^a U_{\nu b}U_{\alpha c}-U_\mu^a V_{\nu b}V_{\alpha c})]\\&+\sin\beta[\partial_\mu U_\nu^a V_\alpha^a+\partial_\mu V_\nu^a U_\alpha^a\\&-\epsilon^{abc}(\tfrac{1}{3}V_\mu^a V_{\nu b}V_{\alpha c}-U_\mu^a U_{\nu b}V_{\alpha c})]\}\end{aligned} \tag{15}$$

where $\mathrm{Re}(A_\mu^a)\equiv U_\mu^a$ and $\mathrm{Im}(A_\mu^a)=V_\mu^a$. Hence for the total SL(2, ℂ) lagrangian, $\mathscr{L}=(\mathscr{L}_{\mathrm{YM}}+\mathscr{L}_{\mathrm{CS}})=\mathrm{Re}(\mathrm{e}^{\mathrm{i}\beta}L)$. In other words the result that the SU(2) complex gauge potentials can be converted to the real SL(2, ℂ) gauge potentials as demonstrated in ref. [6] for the (3+1)-dimensional YM action is also valid for the (2+1)-dimensional theories defined by the YM and CS actions together.

In the last part of this note we exhibit some wave-like solutions to the topologically massive theories. In (3+1)-dimensional YM theories wave solutions are discussed in ref. [8]. Consider the ansatz

$$A_j^a=f^a\delta_j^1 h, \tag{16a}$$

$$A_0^a=-f^a h, \tag{16b}$$

where f^a are functions of $u=(x^1-x^0)$ and h depends on x_2 only. Substituting into the equations of motion (2), we obtain

$$h=l\exp(\xi x_2),\quad l=\text{constant} \tag{17}$$

and f^a are arbitrary functions of u. The field strengths are

$$E_i^a=-f^a\delta_i^2 h', \tag{18a}$$

$$B^a=-f^a h', \tag{18b}$$

where h' indicates differentiation of $h(x_2)$ with respect to x_2. The energy density is given by

$$\theta_{00}=f^2h'^2 \tag{19a}$$

with $f^2=f^a(u)f^a(u)$, and the momentum density is

$$\theta_{0j}=-\delta_i^1 f^2h'^2, \tag{19b}$$

whilst

$$\theta_{ij}=(f^2h'^2)\delta_i^1\delta_j^1. \tag{19c}$$

Note that the energy and momentum densities are equal in magnitude. The solution (16) is an abelian solution and it describes a wave propagating along the x_1-axis direction with amplitude increasing from zero at $x_2=-\infty$ to infinity at $x_2=+\infty$. Incidentally the total action for the solution (16) vanishes.

A nonpropagating wave, i.e. $\theta_{0i}=0$, can also be constructed by using the ansatz

$$A_j^a=\delta_2^a\delta_j^1 P+\delta_j^2(\eta^a Q-\mathrm{i}\xi\delta_2^a), \tag{20a}$$

$$A_0^a=-\delta_2^a P, \tag{20b}$$

where P and Q are arbitrary functions of u and

$\eta^a = (\delta_1^a + \mathrm{i}\delta_3^a)$. The field strengths are

$$E_i^a = \eta^a \delta_i^2 (Q' + \mathrm{i}PQ), \tag{21a}$$

$$B^a = \eta^a (Q' + \mathrm{i}PQ), \tag{21b}$$

with $Q' = \mathrm{d}Q/\mathrm{d}u$. One finds that the total action and the energy–momentum tensor vanish.

Finally we end with some remarks.

(i) That the ansätze (4), (5) and (7) enable us to obtain explicit analytic solutions is because they actually linearize the nonlinear YM equations (2). The solutions obtained are abelian in the sense that $[F_{\mu\nu}, F_{\alpha\beta}] = 0$ although $[A_\mu, A_\nu]$ and $[A_\mu, F_{\alpha\beta}]$ are not zero in general.

(ii) The static solutions given by expressions (7) and (9) are singular at the origin and hence they require the support of a point source at the origin. Very recently, numerical and perturbative finite action solutions have been obtained [9] for topologically massive gauge theories in the presence of external sources J_μ^a. With our ansätze for exact solutions, it is not difficult to derive solutions with prescribed external sources. As an example, consider the ansatz (4) which leads to the solution $C = K_1(z)$ and $H = K_0(z)$ for the YM equations with the CS term, eq. (2). Now add the following external source to eq. (2):

$$J_0^a = -\phi^a [H(s/\rho)^2 + \xi(sC/\rho)], \tag{22a}$$

$$J_j^a = -\phi^a \phi_j (sC/\rho)', \tag{22b}$$

where s is a parameter to describe the total strength of the external source. The following solution is immediate:

$$H = K_s(z), \tag{23a}$$

$$C = K_{s+1}(z). \tag{23b}$$

(iii) The solutions (7) and (9) lead to zero energy–momentum tensor. For comparison we note that in the (3+1)-dimensional YM equations the self-dual solutions also have vanishing energy–momentum tensor.

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