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RAYS AND PHASES IN QUANTUM MECHANICS

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We show how phases enter in quantum mechanics as observable entities, in spite of the fact that physical states are represented by rays; in particular the Berry phase for closed loops and the phase difference between the two paths in the two-slit experiment stem from a common origin.

Key words: phase, closed paths, fiber bundles, interference.

1. The space of states of a quantum mechanical system forms a complex metric projective space, $\overline{\mathcal{H}} = CP(\infty)$, where $\mathcal{H}$ is the Hilbert space of vectors; the ray of the ket vector $|\psi\rangle$ is denoted by ψ. As rays cannot be added, the superposition principle as conventionally stated needs reformulation, and in a recent paper by one of us [1] it is called the decomposition principle, because the metric (scalar product) has the consequence that a given state (ray) ψ has well-defined projections on other rays.

However, one cannot dispense altogether with the vectors $|\psi\rangle$ of the overlying Hilbert space; for example, the very concept of eigenvalue, crucial to the axioms of measurements, is usually stated through a spectral equation involving vectors.

In this paper we set up some simple considerations so as to make the totality {vector space $\mathcal{H}$, projective space $\overline{\mathcal{H}}$} the appropriate frame for the formulation of quantum mechanics. In mathematical

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terms it is the *line bundle* structure which matters:

$$\begin{array}{c} \eta : C^* \to \mathcal{H} - \{0\} \to CP(\infty) \\ \downarrow \\ \ell : C \to E \approx \mathcal{H} \to CP(\infty), \end{array} \tag{1}$$

where $C^* = C - \{0\}$; η is the principal bundle, and ℓ the associated line bundle. The upper line can be replaced by

$$U(1) \to S \to CP(\infty), \tag{2}$$

where $S = S^\infty$ is the injective limit of odd spheres; (2) implies the limitation to vectors of constant norm, in particular the norm 1.

The motion occurs in $CP(\infty)$, but we need the bundle ℓ to precisely state the measurement axioms.

2. The following are some of the reasons why vectors in $\mathcal{H}$ are needed:

(1) Conventional *observables* are linear Hermitian operators acting on $\mathcal{H}$; the spectrum refers to eigenrays or eigensubspaces of $\mathcal{H}$, with a well-defined projection in $\overline{\mathcal{H}}$; there is a number attached to these fixed objects, the corresponding eigenvalue. Notice the invariant character of the spectrum:

$$\text{if} \quad O|\,\psi\,\rangle = o|\,\psi\,\rangle, \qquad O(\lambda|\,\psi\,\rangle) = o(\lambda|\,\psi\,\rangle). \tag{3}$$

(2) The evolution is also defined in the vector (Hilbert) space; that is, the Hamiltonian H, as generator of the motion, is given as a vector field in $\mathcal{H}$; of course it defines also a vector $\overline{H}$ in $\overline{\mathcal{H}}$, which sets the rays (states) in motion. Notice that evolution is also projective invariant, that is, if $|\,\psi(t_0)\,\rangle$ evolves to $|\,\psi(t_1)\,\rangle$, then $\alpha|\,\psi(t_0)\,\rangle$ does to $\alpha|\,\psi(t_1)\,\rangle$.

For the interesting case of *closed* evolution in state space, there is possibly a phase mismatch between the initial and the final vector representatives, which is in principle measurable: If $\psi(t_0) = \psi(t_1)$ indicates the loop in state space, then

$$|\,\psi(t_1)\,\rangle = e^{i\alpha}|\,\psi(t_0)\,\rangle, \tag{4}$$

with a well-defined (total phase) α, which is again projective invariant.

The partition $\alpha = \beta + \gamma$ into a dynamical phase

$$\beta = \int \langle\,\psi\,|\,H\,|\,\psi\,\rangle\,dt$$

and a geometrical (Berry) phase

$$\gamma = \int \langle \widetilde{\psi} \,|\, id \,|\, \widetilde{\psi} \rangle,$$

where $\psi(t) \to |\widetilde{\psi}(t)\rangle$ is a *section* of the line bundle ℓ (cf. [3]), is very often done these days. The uniqueness of the splitting is related to a Hermitian and holomorphic connection in the line bundle ([2, 3]); notice also that α, β, and γ are projective quantities.

(3) More physically speaking, all interference experiments, from Young's double slit (ca. 1800) to neutral kaon decay (1955), seem to appeal to vectors and angles and *a priori* are difficult to understand in terms of rays. Nevertheless, by considering the interference pattern as due to an optical phase difference, we can compare the process with the loop evolution of (2) and relate the angle to the eigenvalue of a (unitary) operator. That the common interpretation of the two-slit experiment is misleading has been also recently shown by the other of us [4]; the possibility of paths encircling the two slits was considered long ago by Biedenharn [5].

3. The central issue we want to emphasize is the *line-bundle* structure; when one represents states as vectors by means of a *section* $\psi: \overline{\mathcal{H}} \to \mathcal{H}$, which is the general concept of wave function; these sections have an equivalence relation

$$\psi \sim \psi' \quad \text{if} \quad \psi'(x) = \lambda(x)\psi(x), \tag{5}$$

with $\lambda(x) \in C^*$. This makes the physics section-independent, and it is a kind of "gauge invariance," first advocated by Weyl (1927) to "incorporate" electromagnetism in wave mechanics. Of course, nothing is lost if $\lambda(x)$ is just a phase, i.e., $|\lambda(x)| = 1$, namely, if we take the principal bundle as being (2); this is so because the evolution, that is to say the flow of a vector field on $E \approx \mathcal{H}$, is unitary.

Let us now see how this extended framework of line bundles helps us to understand the physics; we consider the three previous situations, namely observables and measurement theory, dynamical evolution, and interference phenomena.

(1) Conventional observables do exist and act on vectors; for a state ψ (a ray) and an observable A (a Hermitian operator) the statistical result of a measurement is

$$\exp\,(A[\psi]) = \mathrm{Tr}\; A\pi(\psi), \tag{6}$$

where $\pi(\psi) = |\psi\rangle\langle\psi|/\langle\psi \mid \psi\rangle$ is the ray ψ represented by a trace-one projector. For, (6) includes the eigenstate condition, because, if $A = \Sigma\lambda_a P_a$ is the spectral decomposition of A and if ψ is the eigenstate of A, then $\pi(\psi) \subset P_a$ for some a; in other words, (6) includes the instantaneous and expectation measure of A on state ψ.

(2) The evolution is given by the vector field H acting on $E \approx \mathcal{H}$; therefore

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle, \tag{7}$$

with

$$U(t, t_0) = P \exp\left(-\frac{i}{\hbar}\right) \int_{t_0}^{t} H(\tau)\, d\tau. \tag{8}$$

As we said, this is a projective-invariant evolution; of course, this defines also an evolution in $\overline{\mathcal{H}} = CP(\infty)$, which is the real evolution of states.

Now for closed loops in state space the unitary evolution produces an eigenvalue equation

$$|\psi(t_1)\rangle = U(t_1, t_0)|\psi(t_0)\rangle = e^{i\alpha}|\psi(t_0)\rangle, \tag{9}$$

when $\psi(t_1) = \psi(t_0)$. It is one of the assertions of this paper that *the eigenvalues of unitary operators are as measurable as those of Hermitian operators*, because both are equally projective. In fact the (total) phase α can be and has been measured (e.g., in optical fibers; see [6]); we already remarked on the question of invariant splitting of α into a geometrical part γ and a dynamical part β [7].

4. We come now to consider the two-slit experiment. This crucial experiment (the *only* mystery of quantum mechanics, according to Feynman) takes a quantum particle, e.g., electron or photon through two paths, say C_1 and C_2, and they recombined at a point P on the screen.

We want to compare it with the closed loop evolution mentioned before; namely the loop $C_2^{-1} \circ C_1$, in which the systems goes to the screen at point P through C_1 and returns through C_2; the phase for the closed loop

$$\alpha = \int_{C_1} n(x)\, ds - \int_{C_2} n(x)\, ds \tag{10}$$

$$\exp i\alpha = U(C_2^{-1} \circ C_1) \tag{11}$$

(where nds is the infinitesimal optical path) is clearly what determines the intensity at point P at the screen.

In no way need we say that there is superposition of "states"; there is a unique state, which goes through both paths; several misconceptions in the conventional "explanations" were resolved in [4].

5. In conclusion we would like to stress how the line bundle structure is the right formalism; namely, all the statements are referred to rays and are therefore projective invariant. To obtain them via vectors is legitimate and at times unavoidable, but we must be sure to use them in a gauge invariant way; in particular, phases as obtained in closed loop evolution or in multiple path interference are perfectly observable, and in the bundle setting there is a place for them.

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REFERENCES

1. L. J. Boya, *Found. Phys.* **19**, 1303 (1989).
2. M. Berry, *Proc. Roy. Soc. A* **392**, 45 (1984).
3. A. Bohm, L. J. Boya, and M. Kendrick, "Simple derivation of Berry phase," University of Texas, C. P. T. preprint, March 1990.
4. E. C. G. Sudarshan and T. Rothman, "The two-slit interferometer reexamined," University of Texas, C. P. T. preprint D. O. E.-ER 40200-200, December 1989, to be published.
5. L. C. Biedenharn, private communication to E. C. G. S.
6. A. Tomita and R. Y. Chiao, *Phys. Rev. Lett.* **57**, 937 (1986); E. C. G. Sudarshan, R. Simon, and H. J. Kimble, *Phys. Rev. Lett.* **61**, 19 (1988).
7. This is being investigated by the authors of Ref. 3 above.

NOTES

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