Toward an understanding of the spin-statistics theorem

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We respond to a recent request from Neuenschwander for an elementary proof of the Spin-Statistics Theorem. First, we present a pedagogical discussion of the results for the spin-0 Klein–Gordon field quantized according to Bose–Einstein statistics; and for the spin-$\frac{1}{2}$ Dirac field quantized according to Fermi–Dirac statistics and the Pauli Exclusion Principle. This discussion is intended to make our paper accessible to students familiar with the matrix solution of the quantum harmonic oscillator. Next, we discuss a number of candidate intuitive proofs and conclude that none of them pass muster. The reasons for their shortcomings are fully discussed. Then we discuss an argument, originally suggested by Sudarshan, which proves the theorem with a minimal set of requirements. Although we use Lorentz invariance in a specific and limited part of the argument, we do not need the full complexity of relativistic quantum field theory. Motivated by our particular use of Lorentz invariance, if we are permitted to elevate the conclusion of flavor symmetry (which we explain in the text) to the status of a postulate, one could recast our proof without any explicit relativistic assumptions, and thus make it applicable even in the nonrelativistic context. Such an argument, presented in the text, sheds some light on why it is that the ordinary Schrödinger field, considered strictly in the nonrelativistic context, seems to be quantizable with either statistics. Finally, an argument starting with ordinary-number valued (commuting), and with Grassmann-valued (anticommuting), oscillators shows in a natural way that these must relativistically embed into Klein–Gordon spin-0 and Dirac spin-$\frac{1}{2}$ fields, respectively. In this way, the Spin-Statistics Theorem is understood at the expense of admitting the existence of the simplest Grassmann-valued field.

PART A. INTRODUCTION

This paper is our response to a question raised by Neuenschwander in the "Questions and Answers" section of The American Journal of Physics,[1] whom we quote:

> "In the *Feynman Lectures on Physics*, Richard Feynman said: 'Why is it that particles with half-integral spin are Fermi particles whose amplitudes add with the minus sign, whereas particles with integral spin are Bose particles whose amplitudes add with the positive sign? We apologize for the fact that we cannot give you an elementary explanation. An explanation has been worked out by Pauli from complicated arguments of quantum field theory and relativity. He has shown that the two must necessarily go together, but we have not been able to find a way of reproducing his arguments on an elementary level.... This probably means that we do not have a complete understanding of the fundamental principle involved....'
> Has anyone made any progress toward an 'elementary' argument for the Spin-Statistics Theorem?"

Within months a few responses appeared, none of which we find to be credible.

Neuenschwander's question made us realize that we too did not "really understand" the original Pauli proof,[2] and did not understand the too terse comments on the subject that one finds in all textbooks on quantum mechanics and field theory, especially in the book on the subject by R. F. Streater and A. S. Wightman.[3] Pauli's proof and Streater and Wightman's explanation and expansion of it, are perhaps examples of just the inaccessible formal arguments which Feynman was apologizing for, and which prompted Neuenschwander's dissatisfaction in the first place.

We present the following.

(1) In Part A, we present a pedagogic review of the quantization of the spin-$\frac{1}{2}$ Dirac equation according to Fermi–Dirac statistics and of the spin-0 Klein–Gordon equation according to Bose–Einstein statistics. Following this, we briefly describe important and exciting experimental searches for violations of, for example, the Pauli Exclusion Principle, and theoretical efforts to interpret such violations.

(2) In Part B, a detailed refutation of four heuristic proofs put forward in response to Neuenschwander's question is presented. We are in full accord with a brief criticism already published by Hilborn.[4]

(3) In Part C, a simple proof of the Spin-Statistics Theorem based on work by Schwinger is presented.[5,6] The proof makes use of a convenient but rather unfamiliar formulation of quantum theories which uses second-quantized Hermitian fields rather than the usual complex Schrödinger wave functions. It also uses a Lagrangian formalism with the kinematic part of the Lagrangian bilinear in these fields and, at most, linear in their first derivatives. The proof is based on Sudarshan's observation that rotational invariance, in conjunction with the postulate of flavor symmetry (perhaps as a vestige of Lorentz invariance) of the Lagrangian requires the spin-statistics connection. The flavor symmetry is necessary to prevent a free antisymmetrization on internal degrees of freedom (for example, isospin), which could reverse our conclu-

sions. We will find that flavor symmetry is necessary to satisfy certain elementary requirements of relativistic quantum field theory but seems to have no independent motivation.

We spend considerable effort rationalizing Sudarshan's result, because in this formalism chosen by Schwinger to make the spin-statistics connection transparent, many other things become quite complicated. We describe the situation in some detail. We hope the reader will not be intimidated, and will accept our assurances that the theory, although cumbersome, is truly elementary. All the manipulations involved are familiar from elementary quantum mechanics and classical Lagrangian mechanics. One exception to this claim is the fact that we have used the formalism of second-quantized field operators to deal with the many-body quantum mechanics. We know of no better alternative. The fact that the fields are noncommuting operators can often be kept in the back of one's mind, while they are manipulated quite freely. Only occasionally does one have to confront their full complexity.

(4) In Part D, we introduce a different way of looking at the problem of *understanding* the spin-statistics connection. We argue for the consideration of two fundamental objects. The first is just the well-understood harmonic oscillator. The second is a "Grassmann" oscillator, which is analogous to the ordinary harmonic oscillator but differs in the essential way that the amplitudes of Grassmann oscillators anticommute. Bilinear kinematic Lagrangians for these fundamental objects can be constructed whose symmetry uniquely identifies them by Sudarshan's argument, as integral or half-integral spin in accord with the Spin-Statistics Theorem. Their Lagrangians can be embedded relativistically, and in the simplest cases lead to the Klein–Gordon spin-0 field theory for the ordinary commuting oscillator and to the Dirac spin-$\frac{1}{2}$ field theory for the anticommuting Grassmann oscillator. We characterize this as the *statistics-spin* connection.

Following Part D, we summarize our basic conclusions.

The objection is frequently raised "But why anticommutators?" We are reminded of Rabi asking about the muon, "Who ordered this?" We do not know *why* anticommuting particles are required to exist. We are glad that they do, the point being that none of us would be here, if there was a here, to worry about it if they didn't. But within the confines of the local Lagrangian formulation of relativistic quantum mechanics in ordinary $(3+1)$-dimensional space–time, we do know that they must have half-integral spin, and vice versa. In §C3, by a simple example, we illustrate the apparently *essential* role of relativity in restricting our choice of theories to those with the observed spin-statistics connection.

The Spin-Statistics Theorem is generally credited to Pauli, but Pauli and Weisskopf,[7] Pauli himself,[8] Iwanenko and Socolow,[9] Fierz,[10] Belinfante,[11] Belinfante and Pauli,[12] and deWet[13] all made prior contributions. Pauli did define the terms in which it was proved, and he criticized[14] alternative proposals from Feynman[15] and Schwinger.[16] Finally, Lüders and Zumino,[17] and Burgoyne,[18] based on developments of Hall and Wightman[19] and Jost,[20] resolved flaws of logic, rigor, and generality which infected the earlier proofs. The foundation of the theorem, however, remained the one defined by Pauli: relativistic quantum field theory. We note in passing that in atomic physics, for the conduction electrons in metals, for the phonons in solids, and most recently in Bose–Einstein condensates at tenths of a micro-kelvin, we need the result in a nonrelativistic context.

§A1. Preliminary remarks

Everyone *knows* the Spin-Statistics Theorem but no one *understands* it. This is the complaint expressed by Neuenschwander.

The Spin-Statistics Theorem—which states that *identical half-integral spin particles satisfy the Pauli Exclusion Principle and Fermi–Dirac statistics which permit no more than one particle per quantum state; identical integral spin particles satisfy Bose–Einstein statistics which permits any number of particles in each quantum state*—stands as a fact of nature. The question is whether physics contains this fact as a prediction, and if so how this comes about; or whether physics is merely consistent with the Spin-Statistics Theorem and that some deeper explanation exists.

The situation turns out to be more simple than we had been led to believe. The pre-existing proofs of the Spin-Statistics Theorem have been encumbered by formulations using the full formalism of relativistic quantum field theory. This turns out to be unnecessarily complicated. Following work done by Schwinger and by Sudarshan,[21] the proof of the Spin-Statistics Theorem is reduced to its barest essentials, which are contained in elementary quantum mechanics extended to include Pauli spinors of half-integral spin *supplemented* by one *essential* feature of the relativistic theory.

For the sake of completeness, we present brief, heuristic arguments for the following standard choices:

(1) Anticommutation relations for operators of the generic form

$$[a,b^\dagger]_+ \equiv ab^\dagger + b^\dagger a,$$

leading to antisymmetrized wave functions, Fermi–Dirac statistics, and the Pauli Exclusion Principle for identical particles of spin-$\frac{1}{2}$ satisfying the first-order relativistic free particle Dirac equation.

(2) Commutation relations

$$[a,b^\dagger]_- \equiv ab^\dagger - b^\dagger a,$$

leading to symmetrized wave functions and Bose–Einstein statistics for identical particles of spin-0 which satisfy the second-order relativistic free particle Klein–Gordon equation.

§A2. Anticommutation relations for Dirac spin-$\frac{1}{2}$ fields

The Dirac equation[22] linear in the space and time derivatives is

$$i\frac{\partial}{\partial t}\psi = H_D\psi = \left(\vec{\alpha}\cdot\frac{\vec{\nabla}}{i} + \beta m\right)\psi. \tag{1}$$

Dirac found the four 4×4 matrices $\vec{\alpha}$ and β by requiring that ψ also satisfy the relativistic energy–momentum relation $E^2 = p^2 + m^2$. Consider a state of zero momentum with the Dirac wave function

$$\psi = aue^{-imt} + b^* ve^{+imt}, \tag{2}$$

which requires interpretation[23] of the "negative energy" piece $\sim\exp(imt)$, for which

$$i\frac{\partial}{\partial t}e^{imt} = -me^{imt} \equiv Ee^{imt}.$$

The four-component Dirac column spinors u and v are trivial for zero momentum. Here we use the standard representation with

$$\beta=\mathrm{diag}(1,1,-1,-1).$$

u has the top two components equal to a Pauli spinor, the bottom two equal to zero, and the opposite for v. They are normalized to

$$u^\dagger u=v^\dagger v=1, \quad u^\dagger v=v^\dagger u=0, \tag{3}$$

where $u^\dagger$ is the Hermitian conjugate to u.

We begin by interpreting a (and b) as the amplitude to be in the electron (positron) state with energy $+m$ and charge $e(-e)$ (the charge on the *electron* is taken to be $e=-|e|$. The mass $m=|m|$ is intrinsically positive).

Speaking loosely, ψ has a piece aue^{-imt}, which is the wave function of an electron of energy m, and another piece b^*ve^{imt}, which is the complex conjugate of the wave function of a positron of energy $+m$. The probability for an electron is a^*a, and that for the positron part must be defined in a way that is similar but subject to such basic requirements as positive energy and opposite charge (permitting pair production) for the positron. We are reminded of the raising and lowering operators in the matrix solution of the quantum theory of the harmonic oscillator, where the *operators* a and $a^\dagger$ $(b,b^\dagger)$ replace the *complex numbers* a and a^* (b,b^*) and the number operators $N_e=a^\dagger a$ and $N_{\bar{e}}=b^\dagger b$ replace the corresponding probabilities a^*a and bb^*.

Reinterpreting (a^*,a) as raising and lowering (better, creation and annihilation) operators $(a^\dagger,a)$, we have the following:

(1) $a^\dagger$ creates an electron of energy m, charge e; a annihilates an electron of energy m, charge e. (There is also an unspecified spin component, and usually a momentum which is here set to zero.)
(2) Similarly, $b^\dagger$ creates a positron of energy m, charge $-e$, and so on.

In the usual harmonic oscillator solution of nonrelativistic quantum mechanics, only the operator a and its Hermitian conjugate $a^\dagger$ occur, but not b and $b^\dagger$. There is only one kind of quantum, no charge, and no antiparticle. For the harmonic oscillator case, the operators a and $a^\dagger$ are linear combinations of the coordinate q and momentum p which satisfy canonical commutation relations, and the a and $a^\dagger$ can also be shown to satisfy the commutation relations

$$[a,a^\dagger]_-=1, \quad [a,a]_-=[a^\dagger,a^\dagger]_-=0. \tag{4}$$

In a standard elementary, but fundamental, exercise, the matrix representation of the quantum harmonic oscillator follows from this algebra plus the requirement that the positive definite Hamiltonian operator

$$H=a^\dagger a+\tfrac{1}{2} \tag{5}$$

must have a lowest energy eigenstate $|0\rangle$. We are led to the occupation number operator $N=a^\dagger a$ whose eigenstates $|0\rangle,|1\rangle,\dots,|n\rangle,\dots$, are characterized by the integer occupation number eigenvalues $n=0,1,2,\dots$, which are the number of quanta in each state. To complete the solution we have the orthonormal occupation number eigenstates $|n\rangle$, and the matrix elements

$$\langle n-1|a|n\rangle=\sqrt{n}, \quad \langle n|a^\dagger|n-1\rangle=\sqrt{n}, \quad \langle n|a^\dagger a|n\rangle=n. \tag{6}$$

All others are zero and the energy eigenvalues are

$$E_n-\tfrac{1}{2}=\langle n|H-\tfrac{1}{2}|n\rangle=\langle n|N|n\rangle=n. \tag{7}$$

Returning now to the discussion of the zero momentum solutions of the Dirac equation and the interpretation of the $(a,a^\dagger)$, $(b,b^\dagger)$ as annihilation and creation operators, we can write

$$E=\int \psi^\dagger\left(i\frac{\partial}{\partial t}\right)\psi d^3x=m(a^\dagger a-bb^\dagger). \tag{8}$$

The cross terms disappear because of the orthogonality of u and v. We are now in a quandary. If we attempt to pursue the analogy with the harmonic oscillator solution and assign commutation relations to $(a,a^\dagger)$ and $(b,b^\dagger)$ (which are no longer simply related to coordinate and momentum), then the eigenvalues of $a^\dagger a$ and $b^\dagger b$ (or $bb^\dagger$) would be all positive integers, and the energy eigenvalues would range from $+\infty$ to $-\infty$. There would be no lowest energy state but separate a and b oscillators with positive and negative energies.

The way out of this quandary is to replace commutation relations for the $(a,a^\dagger)$ and $(b,b^\dagger)$ by *anticommutation* relations.[24] It took five years to understand the problem of "negative energy states" in the Dirac equation and to realize that the "filled Dirac negative energy sea" with "holes," should be replaced by the concept of antiparticles. The part of the field operator ψ which $\sim e^{-i\omega t}$ (where we always mean $\omega\equiv+\sqrt{p^2+m^2}$) is to be interpreted as a particle annihilation piece, the part $\sim e^{+i\omega t}$ as an antiparticle creation piece.

With anticommutation relations

$$[a,a^\dagger]_+=[b,b^\dagger]_+=1, \tag{9}$$

the situation clarifies remarkably. It will be necessary to extend these anticommutation relations by requiring all others to be zero,

$$[a,b]_+=[a,a]_+=\text{etc.}=0.$$

With the anticommutation relation for b and $b^\dagger$, the energy

$$E=m(a^\dagger a-bb^\dagger)\Rightarrow m(a^\dagger a+b^\dagger b-1), \tag{10}$$

can be expressed in terms of occupation number operators

$$N_e=a^\dagger a, \quad N_{\bar{e}}=b^\dagger b \tag{11}$$

as

$$E=m(N_e+N_{\bar{e}}-1),$$

and readily interpreted. The energy difference

$$E-E_0=m(N_e+N_{\bar{e}})$$

is positive, as required. The total electric charge and momentum have similar sensible expressions in terms of N_e and $N_{\bar{e}}$.

It is possible to construct a matrix realization of the operator algebra with only two states, $|0\rangle$ and $|1\rangle$, for which

$$a|0\rangle=a^\dagger|1\rangle=0, \quad a^\dagger|0\rangle=|1\rangle, \quad a|1\rangle=|0\rangle. \tag{12}$$

Each state is labeled by its eigenvalue of the occupation number operator $N=a^\dagger a$, so

$$N|1\rangle=1|1\rangle, \quad N|0\rangle=0|0\rangle=0. \tag{13}$$

In matrix form,

$$|0\rangle=\begin{pmatrix}1\\0\end{pmatrix},\quad |1\rangle=\begin{pmatrix}0\\1\end{pmatrix},\quad a=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad a^{\dagger}=\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

and the occupation number operator

$$N=a^{\dagger}a=\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

The eigenvalues of the occupation number matrix, which are the occupation numbers allowed by the anticommuting field operators, are zero and one in accord with the Pauli Exclusion Principle.

One can verify that the wave function for two noninteracting identical particles is antisymmetric in the exchange of the two particles. For this we need the field operator $\Psi^{\dagger}(x)$ which creates a particle at position x and its Hermitian conjugate which annihilates it. We need only the particle a but not the antiparticle b part. We have

$$\Psi(x)=\sum_j a_j\psi_j(x),\quad \Psi^{\dagger}(x)=\sum_j a_j^{\dagger}\psi_j^{*}(x) \tag{14}$$

summed over a complete orthonormal set of single-particle wave functions $\psi_j(x)$, and the corresponding (annihilation) operators a_j. {The notation is standard but confusing. Earlier we used ψ [Eqs. (2) and (8)] to denote a Dirac field *operator*. Here we temporarily use Ψ in order to distinguish the field operator from the single-particle Dirac wave function ψ_j.} To define anticommutation relations independent of the choice of basis states ψ_j, it is necessary that the creation and annihilation operators of different modes should satisfy the prescription that "all others are zero."

The state vector for a particle localized at x is

$$|x\rangle=\Psi^{\dagger}(x)|0\rangle. \tag{15}$$

The amplitude for a particle in state s to be at position x is

$$\langle x|s\rangle=\langle 0|\Psi(x)a_s^{\dagger}|0\rangle=\sum_j \psi_j(x)\langle 0|a_ja_s^{\dagger}|0\rangle$$

$$=\sum_j \psi_j(x)\delta_{js}=\psi_s(x). \tag{16}$$

For a two-particle state,

$$\langle x,y|s,t\rangle=\langle 0|\Psi(x)\Psi(y)a_s^{\dagger}a_t^{\dagger}|0\rangle$$

$$=\sum_{jk}\psi_j(x)\psi_k(y)\langle 0|a_ja_ka_s^{\dagger}a_t^{\dagger}|0\rangle$$

$$=\sum_{jk}\psi_j(x)\psi_k(y)\{\delta_{ks}\delta_{jt}-\delta_{kt}\delta_{js}\}$$

$$=\psi_t(x)\psi_s(y)-\psi_s(x)\psi_t(y), \tag{17}$$

which is antisymmetric as desired.

In summary, the quantization of the Dirac equation using anticommutation relations (instead of canonical commutation relations) for the field operators gives a theory with a positive energy spectrum, which is naturally interpreted in terms of electrons and positrons (of spin-$\frac{1}{2}$, not shown) satisfying the Pauli Exclusion Principle and Fermi–Dirac statistics, and possessing antisymmetric many-particle wave functions.

This situation was in place by 1932. It would occupy some of the best minds in physics over the next generation to satisfy themselves perhaps, but certainly not everyone, that the matter was closed.

Next we look at the comparable situation for the spin-0 relativistic scalar field satisfying the free particle Klein–Gordon equation.

§A3. Commutation relations for Klein–Gordon spin-0 fields

There was a long delay and much confusion before the spin-0 relativistic scalar wave equation was treated in a systematic and rigorous way by Pauli and Weisskopf. In contrast to the Dirac case, everything goes smoothly if we choose commutation relations for the field operators, and the resulting Bose–Einstein statistics for the Klein–Gordon spin-0 particles. If we were to choose anticommutation relations, we would still find a formally positive Hamiltonian, but it is infinite for the vacuum state because

$$E_0=\sum \omega(a^{\dagger}a+bb^{\dagger}).$$

No negative energy problems are generated by the solutions $\sim e^{+imt}$, which still occur just as in the Dirac case, because the Klein–Gordon Hamiltonian is bilinear in the time derivatives and produces a factor $+m^2$ regardless of the sign in the exponential.

The Pauli–Weisskopf quantization of the relativistic scalar field assumes a Lagrangian density which produces the Klein–Gordon equation as the field equation for a complex one-component field $\phi(\vec{x},t)$:

$$\mathscr{L}=\frac{\partial\phi^{\dagger}}{\partial t}\frac{\partial\phi}{\partial t}-\sum_{j=x,y,z}\nabla_j\phi^{\dagger}\nabla_j\phi-m^2\phi^{\dagger}\phi. \tag{18}$$

$\mathscr{L}$ will be Lorentz invariant if ϕ is an invariant scalar under Lorentz transformations. The Hermitian conjugate of the field $\phi^{\dagger}$ anticipates the quantization elevating the fields to operators in the Hilbert space of states.

The generalized momentum canonically conjugate to the fields $\phi,\phi^{\dagger}$ are (abbreviating $\partial\phi/\partial t$ as $\phi_{;t}$ and so on)

$$\Pi_{\phi}=\frac{\partial\mathscr{L}}{\partial\phi_{;t}}=\phi_{;t}^{\dagger},\quad \Pi_{\phi^{\dagger}}=\phi_{;t}. \tag{19}$$

The Euler–Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\phi_{;t}^{\dagger}}+\nabla_j\frac{\partial\mathscr{L}}{\partial\phi_{;j}^{\dagger}}=\frac{\partial\mathscr{L}}{\partial\phi^{\dagger}} \tag{20}$$

is

$$\frac{\partial^2\phi}{\partial t^2}-\nabla^2\phi=-m^2\phi, \tag{21}$$

just the Klein–Gordon relativistic wave equation.

The Hamiltonian density

$$\mathscr{H}=\Pi_{\phi}\phi_{;t}+\phi_{;t}^{\dagger}\Pi_{\phi^{\dagger}}-\mathscr{L}$$

$$=\frac{\partial\phi^{\dagger}}{\partial t}\frac{\partial\phi}{\partial t}+\nabla_j\phi^{\dagger}\nabla_j\phi+m^2\phi^{\dagger}\phi \tag{22}$$

is positive, which guarantees a positive energy spectrum, in contrast to the Dirac case.

One also finds a conserved four-vector charge-current density

$$\vec{j}=e\phi^\dagger \frac{\overleftrightarrow{\nabla}}{i}\phi, \quad \rho=e\phi^\dagger i\frac{\overleftrightarrow{\partial}}{\partial t}\phi. \tag{23}$$

Expressed in terms of momentum eigenstates, the Klein–Gordon field ϕ is

$$\phi=\sum_k \frac{1}{\sqrt{2\omega_k}}\{a_k e^{i(kr-\omega_k t)}+b_k^\dagger e^{-i(kr-\omega_k t)}\}, \tag{24}$$

where we interpret the expansion coefficients as annihilation operators a_k for a particle of momentum $\vec{k}$ and creation operators $b_k^\dagger$ for an antiparticle of momentum $\vec{k}$ and energy $\omega_k=+\sqrt{k^2+m^2}$. Substituting these expansions of ϕ and $\phi^\dagger$ into $E=\int \mathscr{H} d^3x$ and $Q=\int \rho d^3x$ gives the total energy and total charge:

$$E=\sum_k \omega_k(a_k^\dagger a_k+b_k b_k^\dagger), \tag{25}$$

and

$$Q=\sum_k e(a_k^\dagger a_k-b_k b_k^\dagger). \tag{26}$$

All the cross terms cancel, leaving expressions that are readily interpreted in terms of particle and antiparticle occupation numbers

$$N_e=a^\dagger a, \quad N_{\bar{e}}=b^\dagger b, \tag{27}$$

if the operators $a,a^\dagger$ and $b,b^\dagger$ are assigned the commutation relations

$$[a_k,a_{k'}^\dagger]_-=[b_k,b_{k'}^\dagger]_-=\delta_{k,k'} \tag{28}$$

and all others zero.

This leads to Bose–Einstein statistics with occupation numbers for each mode allowed to take any positive integer value, and to symmetric many-particle wave functions.

Without delving into all the pathologies that might develop if we try to quantize Klein–Gordon spin-0 fields with anticommutation relations (see §C3), we present an argument used by Pauli and Weisskopf and by deWet. Although incomplete, it is remarkably close to the proof finally constructed by Lüders and Zumino and by Burgoyne.

For deWet's discussion we need the (anti)commutation relations for the full field operators ϕ and $\phi^\dagger$ at different points of space–time, at equal times, and at coincident points of space–time. For example, for the free fields above

$$[\phi(x,t),\phi(x',t)]_-\sim[a+b^\dagger,a+b^\dagger]_-\equiv 0.$$

This is clear because the commutators involved $[a,a]_-$, $[a,b^\dagger]_-$, and so on, are in the category "all others are zero."

Also, the commutator

$$[\phi(x,t),\phi^\dagger(x',t)]_-=0.$$

This follows, not quite so trivially as above, from direct calculation. It follows also from the classical analog because ϕ and $\phi^\dagger$ are both generalized coordinates q_1, q_2, say, whose Poisson brackets vanish, and whose commutator brackets also vanish. It can also be argued to vanish because these equal time operators are separated by a spacelike interval and the two operations cannot be causally connected, therefore the order of the operation cannot matter. The difficulty with this argument is taking the limit as $x\to x'$ in a convincing way.

One more exercise is worth doing to support our choice of a and b. The equal time commutation relation

$$[\phi(x,t),\Pi_{\phi(x',t)}]_-=[\phi(x,t),\phi_{,t}^\dagger(x',t)]_- \tag{29}$$

can easily be shown from the above expansions in terms of the a, $a^\dagger$'s and b, $b^\dagger$'s to be $i\delta^3(\vec{x}-\vec{x}')$, just the intuitive result for the canonical commutation relation

$$\left[q_k,\frac{1}{i}\frac{\partial}{\partial q_j}\right]_- F(q's)=i\delta_{kj}F(q's). \tag{30}$$

Now we sketch deWet's proof that anticommutation relations are impossible for spin-0 fields.

Suppose we try to replace the canonical commutation by an anticommutation, so we investigate the possibility

$$[\phi(x,t),\phi^\dagger(x',t)]_+\overset{?}{=}0,$$

which we assume to hold at $x\neq x'$. The diagonal matrix element for an arbitrary state $|\mu\rangle$ is

$$\sum_\chi \langle\mu|\phi(x,t)|\chi\rangle\langle\chi|\phi^\dagger(x',t)|\mu\rangle$$
$$+\text{Hermitian conjugate}=0,$$

where we have inserted a complete orthonormal set of states

$$1=\sum_\chi |\chi\rangle\langle\chi|.$$

If this can be continued to $x=x'$, then

$$\sum_\chi |\langle\chi|\phi|\mu\rangle|^2=0,$$

leading to the conclusion that a scalar field operator satisfying the above anticommutation relation has no matrix elements in the Hilbert space and would have to be the null operator.

In this simple and direct way, deWet concluded that the anticommutation relation required $|\phi(x,t)|^2=0$, which could only be satisfied if $\phi(x,t)=0$. This nonoperator statement is the correct conclusion, but does not make explicit the underlying assumptions on the requirements of the Hilbert space.

The Dirac field escapes this fate. The anticommutator $[\psi,\psi^\dagger]_+$ is not in the category "all others are zero," because in the Dirac Lagrangian (linear in the time derivative) $i\psi^\dagger$ is the momentum Π_ψ conjugate to ψ, and this anticommutator, by analogy to the canonical commutators, is not zero, but $\delta^3(x-x')$. The above matrix element is not zero but infinite, $\sim\delta^3(0)$, and the Dirac field survives. deWet went on to show that no such exceptions could exist for tensor fields and therefore that it is impossible to quantize integral spin fields with anticommutation relations. His arguments made essential use of the full Lorentz invariance of the Dirac and the Klein–Gordon Lagrangians.

The shortcomings of these simple proofs and others to be presented over the years were as follows.

(1) Pauli (1940) criticized deWet's proof as limited to spin-0 and spin-$\frac{1}{2}$, and also limited to the canonical formalism which is difficult to carry beyond these low spins be-

cause of the need for a proliferating array of subsidiary conditions. In retrospect this appears as a serious lapse in taste and judgment by Pauli, which caused a long-standing impasse.

(2) None of the proofs (including Pauli's) until those finally put forward by Lüders and Zumino and Burgoyne (1958) included the effect of interactions, but dealt with free fields only, or in the case of Feynman and Schwinger with free fields interacting perturbatively.

(3) The formal requirements underlying the analytic continuations and the manipulations of the usually singular products of operators and their matrix elements were eventually addressed. The formalism reached daunting dimensions in the work of Hall and Wightman, but did substantiate the intuitive conclusions.

Schwinger took an informal view about higher spin particles and about interactions. He maintained that higher spin particles were not fundamental, and assumed that a model using spin-$\frac{1}{2}$ and spin-0 constituents would give the required generalization with sufficient rigor. Schwinger further assumed that perturbation theory could in some way deal with the (non)effect of interactions on statistics.

Schwinger's opinion that spin-0 and -$\frac{1}{2}$ solve the basic problem leaves the truly arduous early work of Fierz, Belinfante, and Pauli himself, based on the spinor representations of the Lorentz group, as not only arcane but also obsolete and unnecessary. We bypass, as well, the beautiful, formal, and economical proofs especially of Burgoyne, which prove the Spin-Statistics Theorem beyond doubt but perhaps also beyond comfortable comprehension, in favor of the proof following Schwinger.

§A4. Alternatives to the standard statistics

Before focusing on the standard choice of Bose–Einstein or Fermi–Dirac statistics, we briefly describe interesting research which has the all-important aim of searching for small violations of these two possibilities. Prototype experiments[25] look for K-shell x-rays from electrons falling into filled atomic orbits, in violation of the Pauli Exclusion Principle. The objection to the experiment by Reines and Sobel within the confines of ordinary quantum mechanics of identical particles is described by Amado and Primakoff,[26] and goes back to the original reasoning which led Heisenberg[27] to the notion of symmetric or antisymmetric many-particle wave functions. As Amado and Primakoff explain again "... the Hamiltonian must treat the identical particles completely symmetrically" and as a result "... identical particles in nonrelativistic quantum mechanics can be described according to *unmixable* symmetry types, and worlds of different symmetry type do not mix." In brief, there are no *small* violations of identity. Nonetheless, Amado and Primakoff conclude that such experiments do have important motivations, one of which is to test the stability of electrons.

Greenberg and Mohapatra[28] have developed a theory which employs a single oscillator with mixed trilinear (anti)commutation relations of the Green type.[29] The theory includes a parameter β which interpolates between Fermi–Dirac statistics ($\beta=0$) and "para-Fermi" statistics ($\beta=1$) with occupation numbers 0, 1, 2 for parafermions in a single state. The result is a parallel world of para-electrons which can violate the usual statistics at order β^2. As described by Greenberg and Mohapatra, the importance of this work is to give a quantum mechanical description of small violations of particle identity and to give a basis for interpreting experiments of the K-shell x-ray search type. They state that the "... theory cannot be represented in a positive-metric (Hilbert) space."

Subsequent developments by Greenberg[30] show that there is an interpretation in the framework of the quantum mechanics of *quons*, based on a generalized *q-mutator* which interpolates continuously between commutator and anticommutator,

$$a_j a_k^\dagger - q a_k^\dagger a_j = \delta_{jk}.$$

Greenberg constructs quon Fock states which include the symmetric, antisymmetric, and mixed symmetry states. The norms of the inappropriate states vanish in the $q=\pm 1$ limit as expected and return the theory to Bose–Einstein or Fermi–Dirac statistics. For other values of q there is now the possibility, for example, of q-electron radiative capture into a symmetric 2 q-electron K-shell orbit from a previously symmetric continuum-bound 2 q-electron state, without violating the requirement of Amado and Primakoff that the Hamiltonian evolution of identical particles not change the symmetry type.

Greenberg also constructs quon number operators and the free-particle Hamiltonian which are generally of infinite order in the a, $a^\dagger$, and lead to operators nonlocal in coordinate space. Greenberg concludes that the criterion of *locality* is of critical importance in limiting the choice of fields to those with standard commutation or anticommutation relations.

In spite of these difficulties, there is profound interest in the general question of violations of the usual statistics. Another possibility referred to by Greenberg and Mohapatra is ordinary quantum mechanics and statistics in a space of $N=3+d$ dimensions. Fermions could have an antisymmetric excitation in the invisible (compactified) extra d dimensions, leaving a symmetric wave function in the observed three dimensions. A quantitative estimate of the effect of such apparent violations of antisymmetry on the Pauli Exclusion Principle tests is obviously very model dependent. Ramberg and Snow interpret their search for a violation of the Pauli Exclusion Principle as an upper bound on Greenberg and Mohapatra's β parameter,

$$\beta^2 \leqslant 1.6\times 10^{-26},$$

which might be compared to the square of the ratio of the electron mass to the Planck mass $\sim 10^{-45}$, or to the GUT mass $\sim 10^{-36}$. More recently, however, membrane theorists have been speculating on a large compactification radius for one of their eleven dimensions, which could give a ratio $\sim 10^{-30}$.

It is clear that these and many other speculations can lead us far afield from our announced goal of "understanding" the Spin-Statistics Theorem. We take an orthodox, even pedestrian, view and restrict the discussion to known, established, elementary, but hopefully fundamental physics. We do this without in any way trying to *inhibit* conjecture, but rather to give a firm basis *for* conjecture by establishing bare minimum postulates which can support the theorem.

Our strategy is: We do ordinary quantum mechanics in ordinary $(3+1)$-dimensional space–time. Within ordinary or standard quantum mechanics we *do* include the technique of second quantization and field theory as a convenient and powerful tool for dealing with the many-body problem. The extension to relativistic quantum field theory, while perhaps intimidating, is *still* the ordinary quantum mechanics of the many-body problem, but with the possibility of transitions between states having different particle numbers.

We define *canonical*, or standard, quantum theory to mean that the theory is derived from a Principle of Least Action based on a local Lagrangian. The original definition of canonical meant the deduction of quantum theory from the prescription of replacing classical Poisson brackets by the corresponding quantum commutator brackets (see again the work of Pauli and Weisskopf in §A3), and the Hamilton–Poisson equations of motion by the Heisenberg-commutator equations. In the presence of anticommutating fields, we must generalize the definition to a purely quantum Principle of Least Action. We retain as much of the structure of classical mechanics as possible, including, for example, the role of the Hamiltonian as the time-translation generator in Heisenberg-commutator equations. The goal of a full quantum dynamics can be achieved at least in simple cases by the prescription of correspondingly simple commutation *or* anticommutation relations. A third alternative *is* possible in a slightly more general interpretation of the Principle of Least Action. This choice is Green's parastatistics, which employs a trilinear commutation relation and has somewhat pathological representations which do not seem to be realized physically. An introduction to the subject can be found in Ref. 29. We will not pursue this branch of the subject here, but refer the reader to an excellent review of modern alternatives by Greenberg, Greenberger, and Greenbergest.[31] The impact on our very traditional view of developments such as color confinement and higher dimension string theory also remains to be explored.

PART B. RESPONSES TO NEUENSCHWANDER'S QUESTION

§B1. Introduction

Neuenschwander's question excited us, obviously, but generated a remarkably limited direct response. Two years after publication of the question, there have been only four published replies.

In our opinion, and in agreement with the response of Hilborn, none of the intuitive arguments put forward constitute satisfactory elementary proofs of the Spin-Statistics Theorem. They leave the situation essentially as Feynman[32] described it 30 years earlier.

§B2. Bacry's proof and Hilborn's critique

We discuss the simplest proof, outlined by Bacry.[33] Bacry identifies an exchange operator $\mathscr{E}$ with a suitably chosen rotation operator R. The situation described in his reply is a particularly simple one which illustrates the essential idea but in fact contains the basic flaw which invalidates his argument from being the sought-after "simple intuitive proof." If his argument had been valid, it would have meant that no reference to relativity was necessary and would have negated all previous ideas on the subject. Later we discuss an apparently more general argument along the same line, appearing in an earlier paper by Broyles,[34] which defines the exchange operator in terms of a rotation operator designed for situations more general than the simple configuration described by Bacry. However, Broyles' argument suffers from the same critical flaw as does Bacry's.

Bacry considers the state of an electron at $(x,y,z) = (+a,0,0)$ with spin component $s_z = +\frac{1}{2}$ described by a wave function

$$\psi_A = \begin{pmatrix} \delta(x-a)\,\delta(y)\,\delta(z) \\ 0 \end{pmatrix}, \tag{31}$$

and another electron at $(x,y,z) = (-a,0,0)$ with $s_z = -\frac{1}{2}$ described by a wave function

$$\psi_B = \begin{pmatrix} 0 \\ \delta(x+a)\,\delta(y)\,\delta(z) \end{pmatrix}.$$

The two-electron wave function is written as

$$\Psi_{AB}(1,2) = \psi_A(1)\,\psi_B(2) \pm \psi_B(1)\,\psi_A(2), \tag{32}$$

where we need to make a choice $\pm$ between a symmetric wave function or an antisymmetric wave function for the two electrons. Under the exchange operation $\mathscr{E}_{12}$ taking $1 \Leftrightarrow 2$,

$$\mathscr{E}_{12}\Psi_{AB}(1,2) \equiv \Psi_{AB}(2,1) = \pm \Psi_{AB}(1,2). \tag{33}$$

Bacry then observes that a finite rotation by π around the y axis leaves this two-particle state unchanged. The rotation is generated by the operator

$$R_y(\pi) = e^{-i\pi J_y} = e^{-i\pi L_y} e^{-i\sigma_y \pi/2} = e^{-i\pi L_y}(-i\sigma_y)$$
$$= e^{-i\pi L_y}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{34}$$

Acting on the wave function ψ_A,

$$R_y(\pi)\,\psi_A(x,y,z) = \begin{pmatrix} 0 \\ \delta(-x-a)\,\delta(y)\,\delta(-z) \end{pmatrix}$$
$$= \psi_B(x,y,z), \tag{35}$$

and on ψ_B,

$$R_y(\pi)\,\psi_B(x,y,z) = -\psi_A(x,y,z).$$

Acting on the two-particle wave function,

$$R_y(\pi)\Psi_{AB}(1,2) = -(\psi_B(1)\,\psi_A(2) \pm \psi_A(1)\,\psi_B(2))$$
$$\equiv -\pm \Psi_{AB}(1,2).$$

Bacry now makes the unjustifiable assumption which negates the proof. From the fact that the two-particle state is invariant under the finite rotation $R_y(\pi)$, he concludes that the wave function is also, and requires that

$$R_y(\pi)\Psi_{AB}(1,2) \equiv \Psi_{AB}(1,2),$$

which, if true, would require the choice of the negative sign in the $\pm$ and would determine that

$$\mathscr{E}_{12}\Psi_{AB}(1,2) \equiv \Psi_{AB}(2,1) = -\Psi_{AB}(1,2),$$

the desired result.

However, the invariance of the state does *not* require the invariance of the wave function in the case of a *discrete* symmetry, which is what we have here. There is nothing to rule out the possibility of a sign change of the wave function under a rotation through π, and therefore the above argument can give no information about the choice of exchange symmetry of the wave function.

The change of sign of a wave function under a discrete symmetry transformation is a common feature of wave functions: Note the invariance under a 2π rotation of a spin-$\frac{1}{2}$

particle state, but the wave function changes sign; or the invariance under reflections of a pseudoscalar state, but again the wave function changes sign.

Broyles' argument, which predates the whole Neuenschwander incident, is primarily concerned with showing that a rotation operator exists which can serve as the exchange operator for two spins not simply parallel or antiparallel (as in Bacry's case), and that therefore a rotation operator exists which exchanges the two particles for general spin states. The actual details of this operator [in Broyles' Eq. (16), a result which is somewhat contrived and artificial in our view] need not concern us. What does concern us is his Postulate A on which he predicates his proof of the Spin-Statistics Theorem:

> "Postulate A: If we write the wave function for two particles in such a way as to exhibit all of the internal quantum numbers and the spatial position of each and, furthermore, if the two sets of quantum numbers including coupling constants and spin are identical with the exception of the spin components along some axis and the spatial positions, *then this wave function must be invariant* (Note: *Italics added*) to any (Poincaré) transformation of the coordinate frame (with all physical apparatus connected to it) that produces a wave function with the same two sets of quantum numbers."

The critical part of this postulate, with which we disagree, is the phrase in italics.

Broyles goes on to emphasize: "Any combination of rotations and translations of the coordinate frame that leaves the picture looking just as it did before these operations, must also leave the wave function unchanged." He does emphasize that the postulate is special to two-particle wave functions for the reason cited above, that the wave function for a single particle with spin-$\frac{1}{2}$ under a rotation of 2π is an immediate exception without this restriction. We conclude that Broyles' Postulate A is *ad hoc* special pleading which has no other purpose than to construct his proof of the Spin-Statistics Theorem, plays no other role, and is to be deleted from the lexicon of quantum mechanics.

In Hilborn's words: Bacry's and similarly Broyles' "... argument establishes a spin-statistics connection at the expense of an additional assumption about how the wave function behaves under coordinate transformations. This assumption goes beyond the requirement that all *observables* remain unchanged and is equivalent to restricting the wave functions to the totally symmetric or totally antisymmetric representations of the permutation group or, equivalently, restricting physical states to those represented by single rays in Hilbert space."

§B3. Topological markers and Feynman's models

The first response to Neuenschwander's question was from Gould,[35] who referred to the Feynman Lectures (1963) and summarized Feynman's 1986 Dirac Lecture[36] on the same topic. Here, as Gould describes it, Feynman "sketched" an elementary argument for the spin-statistics connection. Unfortunately, the fascinating scenarios that he described in his lecture do not constitute a proof or even an explanation of the Spin-Statistics Theorem. In another reply, von Foerster[37] recalled similar heuristic explanations of the result by others.

Gould summarizes Feynman's argument in part, by recounting the paradoxical behavior of the rotation of a tethered classical object. The purpose of the classical paradox is to convince people that a 2π rotation is not just a trivial return of everything to the way it was, even classically, and that we should not be distressed by the resulting change in the sign of the wave function of a spin-$\frac{1}{2}$ particle. The point of the demonstration is—grasping the handle of a full coffee cup, without spilling the coffee, rotate the cup through an angle of 2π around a vertical axis while keeping feet fixed, but at the expense of a twisted arm. A further 2π rotation through a total of 4π returns the cup (and the arm) to the original configuration. This is supposed to remind us of the sign change in the wave function of a spin-$\frac{1}{2}$ particle under a 2π rotation, and the need to rotate the spin-$\frac{1}{2}$ particle twice around through 4π to return to the original wave function. So far, so good. But no further.

Hilborn states it beautifully: "... analogy is not an explanation. Nowhere does the spin of the object enter the discussion nor is it clear what the twist in the constraint has to do with the change in sign of the fermion's wave function. ... why are boson wave functions unchanged... ."

It was Feynman's purpose to show that "... the mysterious minus signs in the behavior of Fermi particles are really due to unnoticed 2π rotations!" Feynman produces in his Dirac Lecture two other models of identical particle exchange which reproduce the spin-statistics connection. One is a nearly classical model of a spin-$\frac{1}{2}$ object which has the required change of sign under a 2π rotation, and also—it is claimed—under an exchange of two identical particles. Feynman describes the composite object invented by Saha,[38] consisting of a spin-0 electric charge e and a spin-0 magnetic monopole of magnetic charge g. The electromagnetic angular momentum

$$\vec{L}=\int \vec{r}\times(\vec{E}\times\vec{B})d^3r \tag{36}$$

is independent of the separation of e and g, directed along the line between them, and equal to eg, giving the Dirac relation $eg=\frac{1}{2}$ when the angular momentum assumes its minimum nonzero value.

Now suppose the electric charge e is moved in a circle around the magnetic charge g. The wave function acquires a phase

$$\phi=e\int \vec{A}\cdot d\vec{l}=e\int \vec{B}\cdot d\vec{\mathscr{S}}. \tag{37}$$

The surface integral $\int d\mathscr{S}$ can be deformed into an easily done integral over a hemisphere centered on the magnetic charge, giving the result

$$\phi=e(4\pi g)/2=eg2\pi=\pi. \tag{38}$$

As desired, the phase of the spin-$\frac{1}{2}$ object has changed by π for a rotation through 2π.

Next, Feynman considers the process of exchanging two (very compact) eg composites, call them 1 at x and 2 at y. He views this as 1 translated from $x\rightarrow y$ in the vector potential of 2, and 2 from $y\rightarrow x$ in that of 1. They are supposed to have their axes parallel and fixed in direction throughout the exchange. Then the phase acquired by the composite wave function during the exchange is

$$\phi=\phi_1+\phi_2=e\int_x^y \vec{A}_2\cdot d\vec{l}_1+e\int_y^x \vec{A}_1\cdot d\vec{l}_2,$$

$$=e\int \vec{B}\cdot d\vec{\mathscr{S}}=\pi, \qquad (39)$$

just the same closed line integral that occurs in the 2π rotation, and just what is required by the Spin-Statistics Theorem.

Finally, Feynman proposes a prescription for the exchange operator $\mathscr{E}_{12}$: Rotate each particle around the other by an angle π, which is equivalent to a 2π rotation of one particle around the other. The net effect for spin-0 particles is a factor 1 and for spin-$\frac{1}{2}$ particles a factor -1, in accord with the Spin-Statistics Theorem.

As a corollary, Feynman reminds us of an argument by Finkelstein[39] to demonstrate that in the rigid rotation of two particles with no rotation of their internal axes, there is a relative rotation of each by π. This can be seen by attaching the two ends of a ribbon, one to each particle, and identifying the "inside edge" initially, which becomes the "outside edge" after the rigid rotation. To complete the exchange and return to the original configuration (including the ribbon) requires a further rotation of each particle around its own body axis by π, for a total rotation by 2π and a factor -1.

§B4. Critique of topological markers

So what *is* wrong with these proofs of the Spin-Statistics Theorem which Feynman sketched?

The argument Feynman borrowed from Finkelstein, which endowed elementary particles with a connecting ribbon to keep track of their orientation, has to be dispensed with for the reason that there are *no* topological appendages to Cartan spinors, as we discuss later in this section. We need a proof that views an elementary particle as a mathematical point or we need to prove that such a view is untenable, but we cannot endow elementary particles with a property which is needed for no other purpose. So a ribbon attached to spinors is not allowed, and we have no reason to identify the exchange operation as a rigid rotation (with internal axes unrotated) followed by two internal rotations through angle π to make the spinors "face" each other again, thereby producing a minus sign. Cartan spinors have no face.

The demonstration based on the charge-monopole composite suffers from the same disqualification of endowing an elementary particle with the unphysical superstructure of a magnetic field. In the ribbon case, the exchange operation is the rigid rotation of each (with internal axes held fixed) followed by two rotations by π which result in the sign change required for Fermi–Dirac statistics. In the composite case, the exchange operation is the rigid rotation of each composite (with internal axes held fixed) but with *no* apparent internal rotations. The purpose of the ribbon model is to convince us that an unintended or unnoticed or at least unmentioned rotation of 2π has actually occurred somehow.

In either case, the topological markers on the elementary particles have to be ruled out as extraneous.

Biedenharn and Louck[40] have discussed just these properties of rotations in Chapter 2 of their book "Angular Momentum in Quantum Mechanics. Theory and Application." They describe in detail and illustrate with intricate diagrams the Dirac construction which demonstrates that "... *for solid bodies* a rotation by 2π is *not* equivalent to the identity, but that a rotation by 4π is... ." The solid (i.e., impenetrable) body is connected to an external coordinate frame by at least three strings which become inextricably tangled after the 2π rotation but can be untangled after 4π. After a number of caveats concerning the Dirac construction, Biedenharn and Louck then make the unequivocal statement "Dirac's result must be carefully distinguished from the similar behavior of spinors under rotation. ... spinors are *point* objects, in contrast to the objects in Dirac's construction, which must have a finite size."

The Cartan definition associates a spinor (ξ_0,ξ_1) with an isotropic vector (x_1,x_2,x_3) defined as a three-dimensional complex Euclidean vector of zero length $x_1^2+x_2^2+x_3^2=0$. The connection is

$$x_1=\xi_0^2-\xi_1^2,\quad x_2=i(\xi_0^2+\xi_1^2),\quad x_3=-2\xi_0\xi_1, \qquad (40)$$

which are nicely expressed using the Pauli matrices as

$$\vec{x}=\xi^T\mathscr{C}\vec{\sigma}\xi, \qquad (41)$$

with $\mathscr{C}=i\sigma_2$. From the bilinear connection between the spinor ξ and the vector $\vec{x}$, Cartan deduced the rotational transformation of the spinor to be the "square-root" of that for the vector. Only the relative sign of ξ_0 and ξ_1 is defined since the spinor reverses sign under a 2π rotation. This is the source of the statement that the Cartan spinor is a point spinor of zero length. It has no other identifying properties beyond its associated isotropic vector which determines two complex spinor components within an overall sign. There is no topological handle, as would be the case for the solid bodies of Dirac's construction and similar classical analogs.

Hilborn's reply refers also to the possibility of the braid group playing a role in understanding the spin-statistics relation. Biedenharn and Louck introduce the braid group of order n as the crossings of n strings attached at top and bottom, which run continuously downward without looping back. There are two elementary operations:

(a) σ_i, which crosses string i *over* string $i+1$ [numbered from the left before (above) the operation];

(b) σ_i^{-1}, which crosses string i *under* string $i+1$.

The operations form a group of operators $\sigma_1,\ldots,\sigma_{n-1}$ for which

(a) σ_0 and σ_1 generate the group through $\sigma_0=\sigma_1\sigma_2\cdots\sigma_{n-1},\quad \sigma_i=(\sigma_0)^{i-1}\sigma_1(\sigma_0)^{-i+1}$,

(b) $\sigma_i\sigma_j=\sigma_j\sigma_i$ for $j\geq i+2$,

(c) $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$.

Biedenharn and Louck characterize the braid group mathematically as more fundamental than the permutation group, and physically as the natural tool to analyze many path-dependent problems. The braid group is different from the permutation group for two dimensions but they coincide for three- or more-dimensional Euclidean space.

Imbo, Imbo, and Sudarshan[41] have shown using braid group analysis that—far from providing an understanding of the spin-statistics connection—topological considerations for point particles lead to a proliferation of presumably unrealized possibilities beyond Bose–Einstein, Fermi–Dirac, and even para and θ statistics. The so-called exotic statistics are associated with higher dimensional representations of the permutation group, in contrast to the Bose–Einstein and the

Fermi–Dirac statistics, which are associated with the one-dimensional totally symmetric and totally antisymmetric representations.

A sophisticated variant of Bacry's, Broyles', and Feynman's identification of an exchange operator as a rotation can be found in the work of Balachandran *et al.*[42] Their work is based on "topological properties of suitable classical configuration spaces and show that the 2π rotation of an individual soliton is homotopic to the exchange of two identical solitons... ." Similar to Bacry's argument, they obtain the spin-statistics connection without relativity or field theory, but with detailed topological assumptions which exclude coincident coordinates from the many-body state space. We can only refer the reader to their detailed arguments, from which we cite some of their qualifying remarks... "one outstanding problem... concerns its relation to field theory... suggestive if as yet vague likeness to Fock space... try to formulate classical Lagrangian mechanics on these spaces and eventually to quantize... ." Certainly, the achievements of this work regarding solitons appear to be impressive, but without a connection to field theory—which is, after all, no more than a convenient and powerful way to handle the quantum mechanics of the many-body problem—its broader significance is still in question.

In Balachandran's theory where topologically nontrivial many-particle manifolds exclude coincident points, and in work based on variable commutation relations, there does not seem to be this canonical foundation which would give hope for a complete dynamical theory. What is referred to there as "field theory" means only that fields can be written down of the form

$$\psi(x)=\sum_j a_j\phi_j(x)+b_j^\dagger\phi_j^\dagger(x),$$

but does not include an Action-Principle-based dynamics for these fields.

Most recently, Berry and Robbins[43] have extended ordinary quantum mechanics to include the physics of exchange by defining an "exchange rotation" operator, by which the spin basis is smoothly parallel transported during exchange. The Pauli sign $(-1)^{2S}$ appears as a geometric phase of topological origin and requires the spin-statistics connection for single valuedness of the wave function. Their work provides a formal basis for the heuristic explanations of Bacry and Broyles. Hopefully, our work can serve as a useful pedagogical introduction to theirs, whose topological complications are certainly not less than the complications of field theory.

§B5. Feynman's unitarity argument

In his 1986 Dirac Lecture, Feynman also returned to an argument for the spin-statistics relation that he had made some 35 years before in his original quantum electrodynamics paper. There he had pointed out that the consistency of perturbative quantum electrodynamics was intimately dependent on a particular sign reversal in the electron–positron one-loop amplitude. The sign reversal can be traced to the feature of the Feynman propagator which replaces "negative energy particles" by positive energy antiparticles. In the second-quantized formulation, the sign reversal arises from the anticommutation $bb^\dagger\rightarrow -b^\dagger b$. The one-loop amplitude is a recurrent feature of many processes and, most fundamentally, it occurs as the first contribution to the vacuum polarization amplitude. Feynman shows that without the sign change, the vacuum-to-vacuum transition probability would be greater than one, and the spin-statistics connection is required in order to avoid this failure of relativistic perturbation theory. Pauli criticized Feynman's analysis and showed that the sign change conjectured for antiparticle amplitudes violated charge-conjugation invariance and amounted to a field theory with an indefinite Hilbert space metric, hence the violations of unitarity manifested as a probability greater than one. In his lecture, Feynman ignored Pauli's comment (to which he evidently never responded) and presented the same argument in a slightly different guise.

Feynman argues that the unitarity of the S matrix requires a cancellation of signs which arise from two sources: One sign change arises from "particles propagating backward in time," requiring two time reversals of a Dirac spinor with a resulting sign change (analogous to that occurring in a 2π rotation); and a cancelling sign change from the anticommutation of Dirac field operators. His claim is that the time-reversal properties of the Dirac spinor require Fermi–Dirac statistics in order to avoid violations of unitarity, clearly containing seeds of Schwinger's proof based on time-reversal invariance, and reversing the logical preeminence of the Spin-Statistics Theorem over the TCP theorem which had been specifically established in the proofs of Lüders and Zumino, and of Burgoyne.

Feynman evaluates the one-loop scattering amplitude using Feynman rules. Consider the following somewhat artificial example, just to make his point: We imagine a toy model of spinless mesons ϕ coupled to Dirac particles ψ by an interaction $\phi^2\bar{\psi}\psi$. The $\phi\phi\rightarrow\phi\phi$ scattering amplitude includes an amplitude with a $\psi,\bar{\psi}$ loop as intermediate state. We are instructed that the same amplitude evaluated with the $\psi,\bar{\psi}$ loop replaced by a loop of two spin-0 particles has the appropriate sign required to respect unitarity.

What is different about the amplitude with the $\psi,\bar{\psi}$ loop? Feynman says that there are two differences. One is a sign change due to the rearrangement—by three anticommutations—of Dirac field operators from the order which occurs naturally in the product of interaction Hamiltonians evaluated at the two vertices

$$\bar{\psi}(2)\psi(2)\bar{\psi}(1)\psi(1),$$

into the order

$$\psi(1)\bar{\psi}(2)\psi(2)\bar{\psi}(1).$$

The rearrangement is necessary so that we can identify the Feynman propagator

$$S_F(2\leftarrow 1)=\langle 0|\psi(2)\bar{\psi}(1)|0\rangle \tag{42}$$

of a particle propagating forward in time from vertex 1 to vertex 2 followed by the Feynman propagator

$$S_F(1\leftarrow 2)=\langle 0|\psi(1)\bar{\psi}(2)|0\rangle$$

of a particle propagating backward in time from 2 to 1. This triple anticommutation results in a characteristic minus sign accompanying every closed fermion loop when evaluated with the Feynman rules.

Minus sign number one. Feynman explains away this minus sign as necessary to cancel another minus sign introduced by a double time reversal of the "actual" antiparticle spinors at vertices 1 and 2 to the "backward in time" negative energy particle spinors. The net result is said to be consistent with the sign of the spinless loop case, which is taken to be consistent with the unitarity of the S matrix.

The projection operator $\Lambda_+(p)$ onto positive energy Dirac states

$$\Lambda_+(p)=\sum_{s=\pm} u(p;s)\bar{u}(p;s)=\frac{p\gamma+m}{2m}, \tag{43}$$

involves free Dirac spinors $u(p)$ satisfying $(p\gamma-m)u(p)=0$ with $p^2=m^2$, $p_0>0$ and normalized to $\bar{u}u=1$. The projection operator onto negative energy states requires an extra minus sign

$$\Lambda_-(p)=-\sum_s u(-p;s)\bar{u}(-p;s),$$

or

$$-\sum_s v(p;s)\bar{v}(p;s). \tag{44}$$

The technical details are reproduced in the Appendix for easy reference.

Minus sign number two. Just Feynman's change of sign from the time reversal $(p\rightarrow -p)$ of two Dirac spinors.

At last, the $\psi,\bar{\psi}$ loop is expressed in a way which makes clear that it has the same sign imaginary part as obtained for a loop with spinless particles. The optical theorem (reviewed in the Appendix), which is based on unitarity, requires that the imaginary part of the forward scattering amplitude should be negative, so all is well. The consistency depends on two minus signs—one from the original anticommutations, one from the double time reflection of the Dirac spinors; and two projection operators, one onto physical, positive energy particle states and one onto so-called "negative energy," but actually physical positive energy antiparticle states.

The perfect equivalence between particle and antiparticle propagation in the Feynman propagator is clear from the original prescription[44] and is discussed in more detail in the Appendix.

The same arguments—in particular, the sign of the one-loop amplitudes—follow also from the path integral formulation of quantum mechanics. The path integral formulation achieves all the results (and more) of canonical quantum field theory without the formalism of second quantization. The device which makes this possible for Fermi–Dirac anticommuting fields is the calculus of *Grassmann* variables, which anticommute with each other. It is a long story, and we simply accept the result that the usual Feynman rules are reproduced.

Commuting Dirac fields violate unitarity as in Feynman's above diagrammatic argument. The interesting questions from this point of view are the basically unanswerable existential question of "Why Grassmann variables?" and a familiar question in reverse (which we finally answer in Part D)—"How do we recognize *a priori* that a Grassmann field is a Dirac field?"

Feynman's argument seems to be a long and tenuous thread on which to hang a proof of the Spin-Statistics Theorem. Any attempt to actually evaluate the severely divergent loop amplitude would lead to even more reluctance to accept this as a fundamental proof of the Spin-Statistics Theorem. We hazard a guess that Feynman's argument leaves Neuenschwander still dissatisfied.

PART C. SCHWINGER AND THE SPIN-STATISTICS CONNECTION

In this section, the long program begun by Schwinger seeking an understanding of the spin-statistics connection is described in its most basic terms. It will be frequently emphasized that Lorentz invariance, or requirements based on it, is still a fundamental necessity.

§C1. Introduction

Schwinger assumed that the kinematic part of the Lagrangian by itself determines the spin-statistics connection. This would be true if weak coupling perturbation theory based on the free Lagrangian was valid. Schwinger claimed the validity of these conclusions independent of perturbation theory on the grounds that at "sufficiently high" energies, the kinematic terms dominate. This is certainly true for potential scattering where the first Born approximation is then valid. It is also true for non-Abelian gauge theories where "asymptotic freedom" makes perturbation theory valid at high energies; but this assertion cannot be taken for granted in all cases.

The validity of Schwinger's assumption is supported in the work of Umezawa and Kamefuchi, of Käll'en, and of Lehmann.[45] They show that a two-point Green's function or propagator for a scalar field

$$F^{(2)}(x-y)=\langle 0|\phi(x)\phi(y)|0\rangle \tag{45}$$

can be expressed as a superposition

$$F^{(2)}(x-y)=\int_0^\infty d(m^2)\rho(m^2)F_0^{(2)}(x-y;m) \tag{46}$$

of the corresponding Green's functions $F_0^{(2)}$ for *free* particles of mass m, with a positive weight function $\rho(m^2)$. A similar conclusion holds for spinor fields. On this basis, it is sufficient to understand the behavior of the free particle Green's function and the commutation properties of the free fields.

Lüders and Zumino, and Burgoyne were able to establish the spin-statistics relation without this assumption within the framework of relativistic quantum field theory obeying certain Wightman axioms. For a scalar field obeying anticommutation relations for spacelike separated field points, it follows from the structure of the Green's function that

$$F^{(2)}(x-y)=0, \quad x-y \ \text{spacelike}.$$

But this implies that

$$\left\| \int dx f(x)\phi(x)|0\rangle \right\|^2=0 \tag{47}$$

for any suitable test function $f(x)$. The conclusion is that the full interacting field $\phi(x)$ annihilates the vacuum and must therefore be a null field. The conclusion is that scalar fields cannot obey the anticommutation relation "all others are zero." Comparable arguments rule out the possibility of commutation relations for spin-$\frac{1}{2}$ fields.

The Wightman-based proofs make no assumptions about equations of motion, and assume no specific Lagrangian; yet the analysis yields conclusions very similar to those obtained by deWet, by Pauli, and by Schwinger. Their common feature is the symmetry properties of the covariants bilinear in the fields, which themselves are finite dimensional representations of the Lorentz group. The relativistic invariance is used only to identify the symmetry or the antisymmetry of such terms. This provides the clue for a much simpler proof, as we see in the following section on Sudarshan's analysis.

§C2. Elementary proof of the spin-statistics theorem

The full complications of Lorentz invariance and relativistic quantum field theory turn out to be unnecessary in Sudarshan's proof of the Spin-Statistics Theorem.[46] They are replaced by a number of intuitive requirements familiar from nonrelativistic quantum mechanics. There still remains—as we will discuss in detail in §C3—a key part of the argument which we have traced only to the requirement of separate relativistic kinematic Lagrangians for individual relativistic fields.

Sudarshan considers a $(3+1)$-dimensional space–time with multicomponent spin wave functions having the usual rotation properties, which are essential to the proof. We impose four conditions on the kinematic part of the Lagrangian for an individual field: It must be

(1) derivable from a local Lorentz invariant field theory for fields which are each a finite dimensional irreducible representation of the Lorentz group (tensor or spinor);
(2) in the Hermitian field basis $\xi=\xi^\dagger$;
(3) at most linear in the first derivatives of the field; and finally, it must be
(4) bilinear in the field ξ.

These conditions impose the requirements that the Euler–Lagrange equations of motion of each basic free field should be first-order, linear differential equations of the Hamiltonian form, local in space and time. The proof makes explicit use only of the rotational invariance guaranteed by the Lorentz invariance.

The *kinematic* terms in the Schwinger Lagrangian have the generic form

$$\mathscr{L}=\frac{i}{2}(\xi_r\dot{\xi}_s-\dot{\xi}_r\xi_s)K^0_{rs}-\frac{i}{2}\sum_{j=1,2,3}(\xi_r\nabla_j\xi_s-\nabla_j\xi_r\xi_s)K^j_{rs}$$
$$-\xi_r\xi_s M_{rs}, \tag{48}$$

summed on indices r,s which are related to the spin of the field, as we will discuss in individual examples. Any "flavor" indices α (corresponding to an internal chargelike degree of freedom) of the field ξ are treated separately, and each such flavor type has its own kinematic Lagrangian. For spin-0 and no flavor, one Hermitian field is sufficient and r, $s=1$; spin $\frac{1}{2}$ and no flavor, r, $s=1,...,4$ (because the Hermitian spinors have two components for the real parts of the Pauli spinors, two more for the imaginary, but must be doubled again for Dirac spinors); spin-0 and one flavor (corresponding to the complex field of a charged particle), r, $s=1$, $\alpha=1,2$ and two separate Lagrangians, $\mathscr{L}^\alpha$; spin-$\frac{1}{2}$ and one flavor, r, $s=1,...,4$, $\alpha=1,2$, and so on. The coefficients K_{rs} and M_{rs} are elements of numerical matrices defined in the (r,s) space of the components of the fields ξ_r. The number of field components (the dimension of the K, M matrices) is left unspecified for the moment, but can be larger than the minimum numbers mentioned above.

Schwinger used Hermitian fields $\xi_r=\xi_r^\dagger$, which are convenient for the special purpose of the spin-statistics connection. The usual complex field of a charged particle requires one flavor and two such Hermitian fields, one for its real part and one for its imaginary part. A familiar example is the pion triplet usually expressed in terms of the charge eigenstates π^+,π^-,π^0 but here in terms of Hermitian fields π_1,π_2,π_3. We write separate kinematic Lagrangians for each Hermitian field, each labeled by a separate flavor index which may be left implicit. Schwinger also required Lorentz invariance and Hamiltonian equations of motion, so he used a Dirac-like Lagrangian at most linear in the first derivatives of the fields. Both of these stratagems are unfamiliar, and can lead to a proliferation of field components in otherwise simpler situations. The simplest case of a real scalar field ϕ satisfying the Klein–Gordon equation requires the introduction of an auxiliary 4-vector field V_μ and 5×5 matrices K and M. We will describe simple examples in detail in the next section, but first we return to Schwinger's Lagrangian and Sudarshan's elementary proof of the Spin-Statistics Theorem.

It is a property of the SO(3) group of proper rotations in three dimensions that representations belonging to *integral spin* have a bilinear scalar (rotationally invariant) product *symmetric* in the indices of the factors: For example, the scalar product of two real vectors is

$$(V_1,V_2)=\sum_{j,k=1,2,3}V_{1j}V_{2k}\delta_{jk}, \tag{49}$$

a familiar result. In contrast, *half-integral spin* representations have *antisymmetric* scalar products: For spin-$\frac{1}{2}$, the scalar product is

$$(\psi_1,\psi_2)=\sum_{r,s=1,2}\psi_{1r}\psi_{2s}(i\sigma_y)_{rs}. \tag{50}$$

This result is familiar from the spin-0 combination of the spin-$\frac{1}{2}$ spinors α and β,

$$\phi_{12}(J=0)=(\alpha_1\beta_2-\beta_1\alpha_2). \tag{51}$$

We note in advance that the invariance of these scalar products under the exchange $1\Leftrightarrow2$ already requires the spin-statistics connection.

The kinematic Lagrangian is of the form

$$\mathscr{L}=\sum_{rs}\xi_r\Lambda_{rs}\xi_s. \tag{52}$$

The matrix Λ_{rs} contains differential operators $\overleftrightarrow{\partial}=\overrightarrow{\partial}-\overleftarrow{\partial}$ as well as numerical Hermitian matrices K,M:

$$\Lambda=\left(\frac{i}{2}K^0\overleftrightarrow{\partial}_t-\frac{i}{2}K_j\overleftrightarrow{\partial}_j-M\right). \tag{53}$$

The terms in the Lagrangian must be scalar invariants under the group of the indices (r,s), that is they must be scalar products bilinear in the ξ_r. The indices r,s are spin indices, and the requirement is rotational invariance of the Lagrangian. A common unsummed flavor index α is implicit on each term in the Lagrangian. The Lagrangian must be invariant under the change of order of any two fields because the order of the fields is undefined *a priori* (within an ignorable c number) and must be irrelevant.

Under the exchange of two fields $\xi_r\Leftrightarrow\xi_s$, the affected terms in the Lagrangian change to

$$\xi_r\Lambda_{rs}\xi_s+\xi_s\Lambda_{sr}\xi_r\rightarrow\pm\xi_s\Lambda_{rs}\xi_r\pm\xi_r\Lambda_{sr}\xi_s \quad \text{(no sum)},$$

with $(+)$ for commuting fields, $(-)$ for anticommuting fields. Invariance of the Lagrangian requires $\Lambda_{sr}=\pm\Lambda_{rs}$. We see from the simplest term in the Lagrangian that the matrix M

must be *symmetric* for *Bose–Einstein* (+) statistics and *antisymmetric* for *Fermi–Dirac* (−) statistics; the opposite holds for the K matrices.

The symmetry type of M and the rotational invariance of the Lagrangian are compatible only with the usual spin-statistics relation: a symmetric scalar product corresponding to an integral spin field for Bose–Einstein statistics with M symmetric; an antisymmetric scalar product corresponding to a half-integral spin field for Fermi–Dirac statistics with M antisymmetric. This is the essential point of Sudarshan's proof.[21,45]

This conclusion is maintained when there is more than one flavor, as Schwinger argued in the following way. Consider the simple case of a complex field ψ whose charge conservation is guaranteed by invariance of the Lagrangian under a global U(1) gauge transformation which changes the phase of ψ by a constant amount

$$\psi \Rightarrow \psi' = e^{i\phi}\psi,$$

corresponding to invariance under rotation by angle ϕ in the two-dimensional flavor space of $\xi^{(1)} = \mathrm{Re}\,\psi$ and $\xi^{(2)} = \mathrm{Im}\,\psi$, and leaving

$$(\xi^{(1)})^2 + (\xi^{(2)})^2 = (\xi'^{(1)})^2 + (\xi'^{(2)})^2.$$

The bilinear kinematic Lagrangian

$$\mathscr{L} = \mathscr{L}^{\alpha=1} + \mathscr{L}^{\alpha=2}$$

has the required gauge invariant flavor singlet behavior

$$\mathscr{L} \sim (\xi^{(1)})^2 + (\xi^{(2)})^2$$

if the K and M matrices are the same in $\mathscr{L}^{(1)}$ and $\mathscr{L}^{(2)}$. Schwinger concludes that the spin-statistics connection can be extended in a gauge invariant way from the basic Hermitian fields to the charged fields.

The result does not explicitly require Lorentz invariance, although it is consistent with Lorentz invariant theories at the expense of doubling (at least) the number of Hermitian field components ξ_r and the dimension of the K and M matrices for a given spin. Three space dimensions are necessary in order to have symmetric or antisymmetric scalar products. A nonrelativistic quantum field theory such as quantum hydrodynamics should be quantized according to Bose–Einstein statistics. By appending a Pauli spin-$\frac{1}{2}$ spinor, we can change the required statistics to Fermi–Dirac, but care must be taken to distinguish spin degrees of freedom from internal degrees of freedom generated by some symmetry group such as isospin. The generators of the spin symmetry must be included in the rotational invariance of the Lagrangian, so the spin indices must be included in the fields ξ_r, in which case their impact on the statistics of the field will be recognized by Sudarshan's theorem. Flavor degrees of freedom such as isospin appear as an overall, unsummed, diagonal flavor index α and have no impact on the spin-statistics connection. In the next section we consider examples that illustrate this requirement of the Schwinger construction and the Sudarshan proof.

§C3. Further comments on the elementary proof

Sudarshan and Schwinger base their proofs on Dirac-like Lagrangians. Both ignore interactions, and both make use of Hermitian fields. Both recognize that the mass matrix M must be symmetric for Bose–Einstein statistics, antisymmetric for Fermi–Dirac. The great simplification in Sudarshan's proof is to recognize that the spin-statistics connection can be made directly from rotational invariance without appealing explicitly—as Schwinger does—to Lorentz invariance and time reversal.

To summarize the preceding section: The Hermitian flavor degrees of freedom α are diagonal in their individual kinematic Lagrangians $\mathscr{L}^\alpha$. The rotationally invariant $\xi^T M \xi$ term in each such Lagrangian has the metric [Clebsch–Gordan coefficient $(sms-m|ss00)$] in the spin space, symmetric for integral spin s, antisymmetric for half-integral spin, and determines the spin-statistics connection for the individual Hermitian fields ξ^α. The Hermitian fields may combine in flavor pairs $(\xi^{(1)},\xi^{(2)})$, which are the real and imaginary parts of complex fields $\psi \neq \psi^\dagger$ which satisfy a global (charge-conserving) gauge invariance under the phase transformation $\psi \to e^{i\alpha}\psi$. Then the pair of Hermitian fields is rigidly rotated by the Hamiltonian leaving the norm $(\xi^{(1)})^2 + (\xi^{(2)})^2$ invariant. The sum of the kinematic Lagrangians $\mathscr{L}^{\alpha=1} + \mathscr{L}^{\alpha=2}$ is left invariant under the gauge transformation. In this way, Sudarshan's proof is extended to include the non-Hermitian field ψ.

Two simple examples are instructive. The isospin-1 pion field charge triplet $\pi^{+,-,0}$ can be written in terms of the three Hermitian components of an isospin vector field $\pi_{1,2,3}$ as

$$\pi^+ = (\pi_1 + i\pi_2)/\sqrt{2}, \quad \pi^- = (\pi_1 - i\pi_2)/\sqrt{2}, \quad \pi^0 = \pi_3,$$

with $\pi^{-\dagger} = \pi^+$ and $\pi^{0\dagger} = \pi^0$. A charge gauge transformation changes

$$\pi^\pm \Rightarrow \pi'^\pm = e^{\pm i\alpha}\pi^\pm,$$

and leaves π^0 unchanged. The effect on the Hermitian fields is to simply rotate the isospin vector fields π_1 and π_2 by an angle α around the 3 axis. A gauge invariant quadratic Hermitian Larangian has the generic form

$$\mathscr{L} \sim \tfrac{1}{2}(\pi^{+\dagger}\pi^+ + \pi^{-\dagger}\pi^- + \pi^0\pi^0),$$

which can be written in terms of the Hermitian components as

$$\mathscr{L} \sim \tfrac{1}{2}(\pi_1^2 + \pi_2^2 + \pi_3^2).$$

The gauge invariance of this form is assured by the invariance of the length of the (1,2) projection of the vector field under a rotation around the 3 axis. We can now choose the parameters of the π_3 part of the Lagrangian to be the same as those of the π_1, π_2 part and get full isospin invariance under rotations around any axis.

A different strategy suggests itself if we ignore the π^0 for a moment. Then another gauge invariant possibility would appear to be

$$\mathscr{L} \sim \tfrac{1}{2}(\pi^{+\dagger}\pi^+ - \pi^{-\dagger}\pi^-) \sim i(\pi_1\pi_2 - \pi_2\pi_1),$$

which is the antisymmetric 3 component of a cross product, also invariant under rotations around the 3 axis. We do not take this possibility seriously here because we have the π_3 field and its flavor symmetric Lagrangian to guide us. Furthermore, we are familiar with the fact that the time-derivative terms in the Klein–Gordon Lagrangian must occur with positive signs to guarantee positive kinetic energy terms in the Hamiltonian, so we are doubly wary of such constructions. A still further objection to such a Lagrangian would be that it is odd under the usual charge conjugation transformation $\pi^+ \Leftrightarrow \pi^-$.

But there is another physical situation where we do not have Hermitian components to guide us—the K-meson isospin doublet K^+, K^0 and its antiparticle doublet $\bar{K}^0$, K^-. Here we consider the impact on the Klein–Gordon field theory of K^+ and K^- mesons of admitting a Lagrangian which is antisymmetric in flavor. The K^+ and K^- fields ϕ^+ and $\phi^- = \phi^{+\dagger}$ satisfy the Klein-Gordon Lagrangian

$$\mathscr{L}_{S/A} = \partial_\mu \phi^{+\dagger} \partial_\mu \phi^+ - m^2 \phi^{+\dagger} \phi^+ \pm (\phi^{+\dagger} \Leftrightarrow \phi^+), \quad (54)$$

which can be written as

$$\mathscr{L}_{S/A} = \mathscr{L}_+ \pm \mathscr{L}_-,$$

symmetric $(S,+)$ or antisymmetric $(A,-)$ in the fields ϕ^+ and ϕ^-. Expressed in terms of Hermitian fields ϕ_1 and ϕ_2 the Lagrangian

$$\mathscr{L}_A \sim i(\phi_1\phi_2 - \phi_2\phi_1).$$

$\mathscr{L}_S$ is the usual theory; $\mathscr{L}_A$ exhibits pathologies which rule it out.

The Euler–Lagrange equation is

$$(1 \pm \mathscr{S})(\partial_\mu^2 + m^2)\phi^+ = 0, \quad (55)$$

where $\mathscr{S} = +1$ for commuting fields required to give a nontrivial result for $\mathscr{L}_S$; and -1 for anticommuting fields required for $\mathscr{L}_A$. The Fourier analysis follows in the usual way as

$$\phi^+ = \phi_+ + \phi_- = \sum_k \frac{1}{\sqrt{2\omega}} \{a_k e^{i(kr-\omega t)} + b_k^\dagger e^{-i(kr-\omega t)}\},$$

where $\omega = +\sqrt{k^2+m^2}$, and the $a, b, a^\dagger, b^\dagger$ satisfy commutation (S) or anticommutation relations (A). The energy and charge follow as

$$E_{S/A} = \sum_k \omega(a_k^\dagger a_k \pm b_k^\dagger b_k), \quad Q_{S/A} = \sum_k q(a_k^\dagger a_k \mp b_k^\dagger b_k).$$

For the usual symmetric case with commuting operators, these are expressed in terms of number operators as

$$E_S = \sum_k \omega(N_k^+ + N_k^-), \quad Q_S = \sum_k q(N_k^+ - N_k^-).$$

For the proposed antisymmetric case with anticommutators and number operators $N^+ = a^\dagger a$ and $N^- = b^\dagger b$,

$$E_A = \sum_k \omega(N_k^+ - N_k^-), \quad Q_A = \sum_k q(N_k^+ + N_k^-).$$

But this is untenable because $b^\dagger$ creates negative charge (as defined by the gauge transformation) and we require a minus N_k^- in Q. The error occurred in jumping to the conclusion that $[b_k, b_k^\dagger]_+ = +\delta_{k,k'}$, as is usually the case. In fact, to satisfy canonical anticommutation relations

$$[\Pi_\phi, \phi]_+ = \delta/i,$$

for a Klein-Gordon field, we need to reverse the sign of the antiparticle anticommutator to

$$[b_k, b_{k'}^\dagger]_+ \to -\delta_{k,k'},$$

which seems impossible. One way to repair the damage is to invoke an indefinite metric in the Hilbert space[47] so $\langle\psi|\psi\rangle = -1$ for b quanta, and identify $N_k^- = -b_k^\dagger b_k = 0,1$. At this price, we return to

$$E_A = \sum_k \omega(N_k^+ + N_k^-), \quad Q_A = \sum_k q(N_k^+ - N_k^-).$$

The usual way to rule out anticommutation relations for Klein–Gordon scalar fields is to invoke relativistic invariance and causality and to require that the effect of two fields at spacelike separations cannot depend on their order of operation, so

$$\langle 0|[\phi(x), \phi^\dagger(y)]_\pm|0\rangle = 0$$

at equal times for all $x - y \neq 0$. In terms of the operators (a,b) the left-hand side is

$$\int \frac{d^3k}{2\omega} (\langle 0|[a,a^\dagger]_\pm|0\rangle e^{-ik(x-y)} + \langle 0|[b^\dagger,b]_\pm|0\rangle e^{+ik(x-y)}).$$

For standard commutators this reduces to

$$\int \frac{d^3k}{2\omega} \sin\{k(x-y)\},$$

which is odd in k and integrates to zero. For standard anticommutators we get the cosine instead of the sine, and a nonzero result in violation of causality. For the pathological anticommutators introduced above, this objection is removed and we pass this test of causality at the price of the indefinite metric.

Another way to rule out anticommutation relations is to recognize that for Klein–Gordon fields, the generalized momentum Π is a field derivative and not a field itself. The canonical anticommutation relations do not specify the anticommutators of the fields themselves, which must be defined by "all others are zero." But this cannot be so because for any state $|\psi\rangle$,

$$\langle\psi|[\phi, \phi^\dagger(x)]_+|\psi\rangle = \sum_\chi |\langle\chi|\phi|\psi\rangle|^2 + |\langle\chi|\phi^\dagger|\psi\rangle|^2 > 0,$$

unless ϕ annihilates all states. This proof also is circumvented by the indefinite metric which would change the relative sign of the two squared matrix elements above. Similarly, charge conjugation must include a metric reversal to restore invariance.

Such pathologies brought on by antisymmetrizing on flavor and thereby reversing the spin-statistics connection have their analogs in Dirac theory. These pathologies are excluded by a basic postulate requiring the Hilbert space to consist of positive energy states with positive definite metric. One consequence is that only the flavor symmetric Lagrangians are admitted into relativistic field theories.[48] It is evident that the pathology is a result of the negative frequency (antiparticle creation) component of the field ϕ_- being inextricably linked to the positive frequency (particle annihilation) component ϕ_+ in the relativistic field ϕ. This linkage is necessary in order to respect proper Lorentz transformations which reverse the sign of the frequency and wave number, Pauli's original "strong reflections." Nothing has been said about the fields ϕ_+ and ϕ_- taken separately, as they are in nonrelativistic theories.

By expressing the relativistic Lagrangians in Schwinger's form, we get Sudarshan's proof based on the rotation subgroup of the full Lorentz group. By elevating the flavor singlet requirement implicit in relativistic theory to a separate postulate, Sudarshan's proof becomes an almost freestanding nonrelativistic proof of the Spin-Statistics Theorem. There is no doubt, however, that the flavor symmetric postulate has deep relativistic roots; nor is there any doubt that there are interesting features—perhaps not always pathological—which occur in prospective counterexamples.

One such counterexample would take a Lagrangian for which the spin statistics has been established and antisymmetrize on particle flavor, for example,

$$\mathscr{L}=\psi^\dagger(i\partial_t-M)\psi\Rightarrow\sum_{j,k=1}^{2}\sigma_y(j,k)\psi_j^\dagger(i\partial_t-M)\psi_k,$$

apparently reversing the original conclusion. However, the Lagrangian can be diagonalized in the flavor indices (j,k) leading to two fields with identical independent Lagrangians of opposite sign, identical but opposite spectra, and a total field energy which is unbounded below.

Next we examine the origin of the sign in the spin metric related to the rotational invariance of the scalar product $\alpha^\dagger\alpha+\beta^\dagger\beta$, which is imposed by defining

$$\alpha_k^\dagger\equiv\alpha_{t/r;k}=-\beta_k,\quad \beta_k^\dagger\equiv\beta_{t/r;k}=+\alpha_k, \tag{56}$$

in terms of the time-reverse spinors. Consistency for all angular momentum requires the spin-metric to be[49]

$$(jmj-m|jj00)=(-)^{j+m}/\sqrt{2j+1}. \tag{57}$$

Two such time reversals result in a phase $(-)^{2j}$ consistent with Feynman's discussion (see the Appendix), and with Schwinger's proofs explicitly using time-reversal invariance. The second-quantized Hermitian field ξ expressed in terms of angular momentum eigenstates U and their time-reverse $U_{t/r}$ is qualitatively given by

$$\xi\sim\sum aU+b^\dagger U^\dagger\rightarrow\sum aU+a^\dagger U_{t/r}. \tag{58}$$

The one-particle expectation value of the mass term in the Lagrangian is

$$\langle 1|L\sim\xi^T M\xi|1\rangle\sim\sum U_{t/r}U. \tag{59}$$

Summed over spin components it is a rotational invariant as required.

The nonrelativistic Schrödinger equation for the complex wave function ψ can be included in the proof by taking the nonrelativistic limit of the Klein–Gordon Lagrangian in the case of integral spin, and the Dirac Lagrangian for half-integral spin. Without using the limit of the relativistic theory, the nonrelativistic Schrödinger theory can evidently be quantized with either Bose–Einstein or Fermi–Dirac statistics. The difficulty appears because the Schrödinger Lagrangian is not directly of the Schwinger form, but contains terms analogous to $\phi_1\phi_2$. Without additional arguments, we cannot rule out the possibility of antisymmetrizing such terms, leaving open the possibility of either choice of statistics. This leads to an implicit but critical reliance on relativistic wave equations.

The electromagnetic field is an almost trivial case using Sudarshan's proof. Since the electromagnetic field is Hermitian, it can be understood without recourse to the Schwinger Dirac-like Lagrangian. The behavior of the electric field term

$$E^2=\sum_{r,s=1,2,3}\partial_t A_r\partial_t A_s g_{rs}$$

in the Lagrangian (or the Hamiltonian) is sufficient to require Bose–Einstein quantization using Sudarshan's argument. Rotational invariance requires $g_{rs}=g_{sr}$ to be symmetric; invariance of the Lagrangian (within a c number) under the exchange $A_r\Leftrightarrow A_s$ requires commutation and therefore Bose–Einstein statistics.

None of the above considerations limit the statistics of composite and nonlocal entities. But usually a composite particle can be considered as a collection of point particles with exchange symmetry simply the product of those of the constituents. When topological obstructions to simple exchange occur, the situation can be more complicated,[50] as occurs in the charged particle-magnetic monopole system of Saha, in the case of charged particles with a minimal Chern-Simons interaction in $(2+1)$-dimensional space–time, and in the Skyrme model of the nucleon as a topological knot in a spin-0 pion field, but having many of the properties of the nucleon. None are simple additive-multiplicative constituent models.

The explicit construction for relativistic anticommuting fields of higher half-integral spin turns out to be impossible. The relativistic Hermitian spinors have extra dimensions (from 12 to 24 depending on the system used, in the case of spin-$\frac{3}{2}$) which must be reduced to $2(2S+1)$ $(=8)$ independent degrees of freedom by subsidiary conditions. Johnson and Sudarshan[51] show that this program is blocked by the appearance of anticommutation relations of indefinite sign where positive definite ones are required. In the anticommutation relations

$$[\xi_j,\xi_k]_+=\sim\delta_{jk}\Rightarrow K_0^{-1}{}_{jk},$$

the matrix K_0 is usually either indefinite or singular, indicating that the fields ξ_j are not independent. Projection operators must be found, a sequence of constraints imposed, and nonsingular K matrices of reduced dimension constructed. But this program is halted by the fact that the reduced K matrix is dependent on the field couplings except in the case of spin-$\frac{1}{2}$, where no constraints are necessary and $K=1$. As a result, for a charged spin-$\frac{3}{2}$ field the anticommutator depends on the external field in such a way that the quantization becomes inconsistent.

Johnson and Sudarshan conclude that only spin-0, spin-1, and spin-$\frac{1}{2}$ fields can be regarded as fundamental. Higher spin fields must be composite and cannot be represented by a local action principle.[52] Their result supports the view that the spin-statistics connection need be demonstrated only for spin-0 and spin-$\frac{1}{2}$.

PART D. UNDERSTANDING THE SPIN-STATISTICS CONNECTION

§D1. Dirac equation from Grassmann theory

Finally, and most simply, we deduce the spin-statistics connection *starting* from Grassmann variables[53] *defined* by the fundamental anticommutation relation

$$\xi_j\xi_k+\xi_k\xi_j=\delta_{jk}. \tag{60}$$

By a series of inferences, we show that the only possible Lagrangian for the associated field is a first-order Dirac Lagrangian. From this point the Dirac equation with 4-component spinors, spin-$\frac{1}{2}$, and all the rest follows as usual. The difference is that we have *started* with an anticommuting quantum field which at the outset was required to satisfy the Pauli Exclusion Principle, and—by inference from the *only possible* Lagrangian for anticommuting objects—must satisfy the Dirac equation.

Schwinger's Lagrangian, linear in the first derivative of the field, suggests that we start with a Grassman variable defined at a point, a function of time only, and construct the basic dynamics. By embedding the result into a Lorentz invariant form, we limit the possibilities to a Grassmann field of spin-$\frac{1}{2}$ satisfying the Dirac equation. This program starts with the prescription

$$\mathscr{L}=\frac{i}{2}\sum_k \xi_k\dot{\xi}_k-\mathscr{H}_I(\xi), \tag{61}$$

where the interaction Hamiltonian $\mathscr{H}_I$ will be specified in a moment. Subject to certain restrictions which we discuss in the following sections, this form turns out to be unique. The kinetic term $(i/2)\xi^T\dot{\xi}$ is Hermitian provided the ξ's anticommute and are themselves Hermitian. The simplest choice is a two-component object with $k=1,2$. The generalized momentum

$$\Pi_k=\frac{\partial\mathscr{L}}{\partial\dot{\xi}_k}=\frac{i}{2}\,\xi_k \tag{62}$$

(by convention, all derivatives on anticommuting objects are from the right) leads to the anticommutation relations

$$[\xi_k,\Pi_j]_+=\frac{i}{2}\,\delta_{kj}\Rightarrow[\xi_k,\xi_j]_+=\delta_{kj}. \tag{63}$$

This is positive definite as required. The prescription "all others are zero" will be invoked where needed. The anticommutation relation is canonical except for the factor $\frac{1}{2}$, which is necessary for the Hamilton equations of motion to agree with the Euler–Lagrange equations.

The Hamiltonian

$$\mathscr{H}=\Pi_k\dot{\xi}_k-\mathscr{L}=\mathscr{H}_I(\xi), \tag{64}$$

where the term linear in the velocity $\dot{\xi}$ disappears as usual. In the kinematic Lagrangian, we choose

$$\mathscr{H}_I=\tfrac{1}{2}\xi^TM\xi, \tag{65}$$

with $M^\dagger=M=-M^T$ in order that $\mathscr{H}^\dagger=\mathscr{H}=\mathscr{H}^T$. The Euler–Lagrange equation leads to $\dot{\xi}_k=-iM_{kh}\xi_h$. The same result follows from the Hamilton equation of motion $\dot{\xi}_k=i[\xi_k,\mathscr{H}_I]_-$ using the anticommutation relations above.

The specific 2×2 example we discuss will have

$$M=\begin{pmatrix}0 & -i\\ +i & 0\end{pmatrix}.$$

If the single oscillator is extended to a field in three space dimensions, we can already identify the two component Grassmann variable as spin-$\frac{1}{2}$. This follows directly from the required rotational invariance of the interaction Hamiltonian $\mathscr{H}_I$, as discussed in §C2, and the antisymmetry of M. If we were to assume a three-component Grassmann variable, we could *not* embed it in a rotationally invariant theory. The M matrix would be antisymmetric, but could not be the spin metric for a half-integral spin particle which must have even dimension. According to the proof of Johnson and Sudarshan, four and more component Grassmann objects must lead to frustrated theories also.

Next we observe that the Lagrangian $\mathscr{L}$ contains only a first derivative and already invites a Dirac embedding. There is a minor technicality. So far the fields have been Hermitian which puts the Dirac equation into the Majorana representation[54] where, for example, $\gamma_0=\gamma_0^\dagger=-\gamma_0^T$ similar to M but 4×4. We can infer

$$\mathscr{L}_D=\frac{1}{2}\int d^3x\,\psi^T(x,t)\gamma_0(\gamma_0E-\vec{\gamma}\cdot\vec{p}-m)\psi(x,t). \tag{66}$$

Here $\gamma_j=-\gamma_j^\dagger=\gamma_j^T$ for $j=x,y,z$ in the Majorana representation. Having made the embedding, we can go to a general representation with complex Dirac spinors by a unitary transformation and return if we wish to the familiar standard representation. We have generalized the summation on k to include an integration over the spatial positions so that the anticommutation relations are generalized to $\sim\delta_{jk}\delta^3(x-x')$ when we invoke the "all others are zero" prescription. Also the two two-dimensional Grassmann variables ξ_k defined independently at each point and satisfying a pointlike Schrödinger equation have been embedded in an irreducible (that is, not separable) way into a covariant structure of four dimensions. This is done in the Schwinger notation of §C2 by combining the two-component Grassmann variables ξ,ξ' to a four-component ψ,

$$\psi=\begin{pmatrix}\xi\\ \xi'\end{pmatrix},$$

and the K^0 and M matrices to (subscripts 2,4 are the dimension)

$$K_4^0=\begin{pmatrix}K_2^0 & 0\\ 0 & K_2^0\end{pmatrix}=\begin{pmatrix}1_2 & 0\\ 0 & 1_2\end{pmatrix}\equiv 1_4=\gamma_0^2,$$

the coefficient of E in Eq. (66); but, for irreducibility,

$$M_4=\begin{pmatrix}0 & M_2\\ M_2 & 0\end{pmatrix}=\begin{pmatrix}0 & \sigma_y\\ \sigma_y & 0\end{pmatrix}\equiv\gamma_0,$$

the coefficient of m in Eq. (66). The last step in the embedding is to include the space derivatives in the Dirac way to obtain a covariant structure.

The *second-quantized* Dirac field follows immediately with all its consequences, including the Dirac wave function and the identification of the spin-$\frac{1}{2}$.

§D2. No Bose–Einstein Dirac equation

The question is whether there can be a commuting field which has a Dirac Lagrangian. Clearly, in a sense, there can. We have mentioned that the Klein–Gordon equation for a relativistic scalar Hermitian field requires 5×5 K and M matrices, a scalar field ϕ, and an auxiliary four vector field V_μ. What is eliminated by Dirac's classic development is the possibility of elementary representations satisfying a Hamiltonian equation with only a first time derivative, without auxiliary fields. In this case, the Lagrangian (for a particle at rest)

$$\mathscr{L}=\frac{i}{2}(\xi^TK\dot{\xi})-\frac{1}{2}\,\xi^TM\xi \tag{67}$$

must have $K=K^\dagger=-K^T$ imaginary antisymmetric and $M=M^\dagger=M^T$ real symmetric in order to give nonvanishing terms in the Euler–Lagrange equations of motion. The generalized momentum

$$\Pi_k = \frac{\partial \mathscr{L}}{\partial \dot{\xi}_k} = \frac{i}{2}\, \xi_s K_{sk} \tag{68}$$

and the canonical commutation relations become

$$[\xi_r, \Pi_s]_- = \frac{i}{2}\, \delta_{rs} \Rightarrow [\xi_r, \xi_s]_- = K_{rs}^{-1}. \tag{69}$$

Now we look for various finite dimensional representations. We easily see that there can be no one-dimensional representation because K is antisymmetric with no diagonal element. For a two-dimensional representation, $K = \sigma_y$. When we try to embed the theory in three-dimensional space and require it to be relativistically invariant we must eliminate the two-dimensional representation on the now familiar grounds of Dirac's algebra. A candidate K matrix for a three-dimensional representation is L_y, which satisfies $L_y = L_y^\dagger = -L_y^T$ but has no inverse, and fails to give sensible commutation relations for the fields, as well as failing the Dirac algebra.

It seems impossible for a Bose–Einstein field to have a Dirac Lagrangian but we continue to explore the situation. For this we turn to the spectrum of the Hamiltonian for representations which can be embedded relativistically. The Hamiltonian is simply

$$\mathscr{H} = \Pi^T \dot{\xi} - \mathscr{L} = \tfrac{1}{2} \xi^T M \xi$$

with M real symmetric. The trivial choice $M = 1$, the unit matrix, corresponds to a product of all covariant or all contravariant representations of the group O(4) which cannot be embedded into representations of the Lorentz group. In order to get products of covariant and contravariant representations, of dimension at least 4×4, we must choose a nontrivial $M = M^\dagger = M^T$ such as

$$M = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}.$$

But this M has equal numbers of $+1$ and -1 eigenvalues and corresponds to a Hamiltonian with negative energies. These are just the signs that are reversed by the anticommuting Grassmann variables in the allowed Grassmann–Dirac theory, and serve to eliminate the possibility of the relativistic embedding of a Bose–Einstein Dirac-like equation.

§D3. No Fermi–Dirac Klein–Gordon Lagrangian

A different Lagrangian which we might consider for the Grassmann variables is the Klein–Gordon form

$$\mathscr{L} = \tfrac{1}{2} \dot{\xi}^T K \dot{\xi} - \tfrac{1}{2} \xi^T M \xi. \tag{70}$$

Now $K^\dagger = K = -K^T$ and $M^\dagger = M = -M^T$ are required for $\mathscr{L}^\dagger = \mathscr{L} = \mathscr{L}^T$. The generalized momentum

$$\Pi_s = \frac{\partial \mathscr{L}}{\partial \dot{\xi}_s} = -K_{sr} \dot{\xi}_r = \dot{\xi}_r K_{rs} \tag{71}$$

defines the canonical anticommutator

$$[\xi_s, \Pi_r]_+ = i \delta_{sr} \Rightarrow [\xi_s, \dot{\xi}_r]_+ = i K_{sr}^{-1}. \tag{72}$$

The usual prescription "all others are zero" includes

$$[\xi_s, \xi_s]_+ = 0$$

and would require $\xi_s \equiv 0$. The conclusion is that Grassmann fields cannot have a Klein–Gordon Lagrangian. The conclusion is somewhat trivial for one-dimensional Grassmann variables which do not exist anyway. The prescription "all others are zero" which requires $\xi_s = 0$ may not seem particularly well founded, but without it the Hamilton equations of motion are not defined and we lose the Hamiltonian as the generator of time translations, and with it the powerful structures of classical mechanics. If a dynamical structure parallel to the canonical field theory of Bose–Einstein fields is to exist for Fermi–Dirac fields, "canonical" anticommutation relations seem to be essential. If we accept that, then the Grassmann Lagrangian of §D1 is unique as is the embedding in the Dirac Lagrangian.

Pressing on with the Euler–Lagrange equations, a nontrivial 4×4 choice for M and K is

$$M = \begin{pmatrix} 0 & i\sigma_x \\ -i\sigma_x & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i\sigma_z \\ -i\sigma_z & 0 \end{pmatrix},$$

giving

$$K^{-1} M = \begin{pmatrix} i\sigma_y & 0 \\ 0 & i\sigma_y \end{pmatrix}.$$

Solutions of the form $\xi(t) \sim e^{\pm i\omega t}$ exist only if the eigenvalues ω^2 of $K^{-1}M$ are real and positive. This is clearly not the case in our example where they are $\pm i$. Other choices for M and K either lead to the same result or are the trivial choice $K^{-1}M = 1$ corresponding to the direct product of two decoupled 2×2 representations which cannot be relativistically embedded.

We conclude that no relativistic Grassmann field can be a Klein–Gordon field. Of course the Grassmann field does satisfy the Klein–Gordon equation as a result of the Dirac equation, but that is distinct from having a Klein–Gordon Lagrangian rather than a Dirac Lagrangian.

PART E: CONCLUDING REMARKS

Sudarshan's arguments based on rotational invariance lead to a simple, transparent, and elementary proof of the Spin-Statistics Theorem, greatly simplifying a proof due to Schwinger based on time-reversal invariance. Sudarshan's proof eliminates the explicit dependence of the proof on relativistic quantum field theory. A critical implicit dependence on relativity is still present, however, as described in §C3.

A fundamental understanding of the spin-statistics connection is obtained in the derivation of the Dirac equation as the only possible relativistic embedding of the Lagrangian theory of the simplest point Grassmann oscillator. The basic field is defined at the outset as an anticommuting quantum field and, by deWet's arguments, is found to satisfy the Dirac equation for spin-$\frac{1}{2}$. The arguments of Johnson and Sudarshan rule out the possibility of fundamental fields having half-integral spin greater than $\frac{1}{2}$, so the fundamental connection between Grassmann variables and Dirac spinors is established. Schwinger's arguments for composite fields are sufficient in other cases. The Klein–Gordon Lagrangian with canonical anticommutation relations is ruled out for anticommuting Grassmann fields by analogs of deWet's theorem. Intrinsically positive anticommutators turn out to be negative. Commuting Bose–Einstein fields cannot have a Dirac Lagrangian, which would lead to negative energies.

Understanding the puzzle of the spin-statistics connection requires that we admit the existence of the most elementary (two-component) Grassmann oscillators, which anticommute and must relativistically embed in the spin-$\frac{1}{2}$ Dirac equation. Commuting fields cannot satisfy the Dirac Lagrangian and relativity and have a positive definite Hamiltonian, an old result. Conversely, a Klein–Gordon Lagrangian for an anticommuting field leads to null fields, another old result. Commuting fields satisfy the Klein–Gordon Lagrangian without contradiction, again an old and familiar result.

Clearly, a unifying point of view for understanding the spin-statistics connection presents itself. Start with two fundamental oscillator fields: a commuting one, which must have a Klein–Gordon Lagrangian and spin-0; and an anticommuting one, which must have a Dirac Lagrangian and spin-$\frac{1}{2}$.

In summary, we have simplified the problem in two steps. The first step is Sudarshan's demonstration that the rotational invariance of the Lagrangian requires the Spin-Statistics Theorem in a simple way, which however does still depend on relativistic quantum field theory for a key argument. In the second step we make the *spin-statistics* connection *understandable* by reversing the question to that of the *statistics-spin* connection. We show that ordinary classical commuting Bose–Einstein number-valued oscillators embed naturally into relativistic quantum field theoretic Klein–Gordon fields of spin-0; not-so-ordinary anticommuting Fermi–Dirac Grassmann-valued oscillators embed naturally into relativistic quantum field theoretic Dirac fields with spin-$\frac{1}{2}$. What remains to be understood in more fundamental terms is the existence of the two types of oscillator: number valued and Grassmann valued.

Finally we are forced to conclude that although the Spin-Statistics Theorem is simply stated, it is by no means simply understood or simply proved.

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APPENDIX: TECHNICAL DETAILS OF FEYNMAN'S UNITARITY ARGUMENT

First Feynman establishes that spin-$\frac{1}{2}$ states change sign under two time reversals. The effect of time-reversal T on spin-$\frac{1}{2}$ states must be

$$T|m_z=+\rangle=e^{i\phi}|m_z=-\rangle,\quad T|m_z=-\rangle=e^{i\xi}|m_z=+\rangle. \tag{A1}$$

With $TF=F^*T$ defining the effect of weak time reversal on algebraic functions F to be complex conjugation, two time reversals give

$$T^2|m_z=+\rangle=e^{i(\xi-\phi)}|m_z=+\rangle. \tag{A2}$$

Feynman shows that we cannot choose $\phi=\xi$. For consider the time reversal of states quantized along the x axis. With

$$|m_x=\pm\rangle=|m_z=+\rangle\pm|m_z=-\rangle, \tag{A3}$$

then

$$T|m_x=+\rangle=e^{i\psi}|m_x=-\rangle=e^{i\psi}(|m_z=+\rangle-|m_z=-\rangle)$$
$$=e^{i\phi}|m_z=-\rangle+e^{i\xi}|m_z=+\rangle,$$

so we must have

$$e^{i\psi}=e^{i\phi}\quad\text{but}\quad -e^{i\psi}=e^{i\xi}$$

and

$$e^{i(\xi-\phi)}=-1.$$

The result is that two time reversals change the sign of a spin-$\frac{1}{2}$ state, and, by superposition, any half-integral spin state. The sign reversal does not occur for integral spin states because they include the unique $M=0$ state for which

$$T^2|M=0\rangle=Te^{i\alpha}|M=0\rangle=e^{-i\alpha}T|M=0\rangle$$
$$=e^{-i\alpha}e^{+i\alpha}|M=0\rangle,$$

so $T^2=1$, a result that can be extended to all integral spin states because they are superposable with the $M=0$ state.

Next, Feynman considers the unitarity of the scattering matrix S, the operator which evolves the quantum system from early (noninteracting) times, through the scattering interval, to late (again noninteracting) times

$$\Psi(t\to\infty)=S\Psi(t\to-\infty). \tag{A4}$$

In order to maintain orthonormality and completeness of states propagated through the scattering, the S matrix must be unitary,

$$S^\dagger S=SS^\dagger=1.$$

In terms of the transition matrix $\mathscr{T}$,

$$S=1-2i\mathscr{T}. \tag{A5}$$

These matrix operators are familiar in their elementary form for individual partial waves elastically scattered by a central potential. In this case, $S=e^{2i\delta}$, $\mathscr{T}=-e^{i\delta}\sin\delta$, and the cross section is $|\mathscr{T}|^2=\sin^2\delta$ within factors of no concern here. The unitarity of the S matrix imposes a requirement on the $\mathscr{T}$ matrix,

$$S^\dagger S=(1+2i\mathscr{T}^\dagger)(1-2i\mathscr{T})=1-2i(\mathscr{T}-\mathscr{T}^\dagger)+4\mathscr{T}^\dagger\mathscr{T}.$$

For diagonal matrix elements $\mathscr{T}_{ii}$,

$$\text{Im}\,\mathscr{T}_{ii}=-(\mathscr{T}^\dagger\mathscr{T})_{ii}=-\sum_j|\mathscr{T}_{ji}|^2\leq 0. \tag{A6}$$

For a given state i, the sum is over all energy and momentum conserving states j. The right-hand side has a ready interpretation in terms of the total cross section for scattering from state i to all possible states j, and it unequivocally determines the sign of the imaginary part of the diagonal $\mathscr{T}$-matrix elements, which correspond to forward elastic scattering. These results are familiar for the scattering amplitudes of individual partial waves elastically scattered by a central potential. There, $-\text{Im}\,\mathscr{T}=\sin^2\delta=|\mathscr{T}|^2=\sigma/4\pi k^2$ with σ the partial cross section, and k the momentum.

The Feynman propagator

$$S_F(x_2-x_1)=-i\int \frac{d^3p}{(2\pi)^3}\frac{m}{E}[\Theta(t_2-t_1)\Lambda_+(p) \times e^{-ip(x_2-x_1)}+\Theta(t_1-t_2)\Lambda_-(p)e^{+ip(x_2-x_1)}] \quad (A7)$$

is the amplitude for a free Dirac particle to propagate forward in time from $1\rightarrow 2$ when the unit step function $\Theta(t_2-t_1)=1$ or for a free Dirac antiparticle to propagate forward in time from $2\rightarrow 1$ when the other step function $\Theta(t_1-t_2)=1$. It is the great—but not entirely free—elegance of the Feynman Rules to treat the antiparticle propagation as if it were negative energy particles propagating backward in time.

The projection operator $\Lambda_+(p)$ onto positive energy Dirac states, in terms of free Dirac spinors $u(p)$ satisfying $(p\gamma-m)u(p)=0$ with $p^2=m^2$, $p_0>0$ and normalized to $\bar{u}u=1$, is

$$\Lambda_+(p)=\sum_{s=\pm} u(p;s)\bar{u}(p;s)=\frac{p\gamma+m}{2m}. \quad (A8)$$

We begin to understand Feynman's argument. It becomes clear that the Feynman Dirac propagator on the mass shell, where the virtual particles become real and contribute to the unitarity sum in the imaginary part of the amplitudes, is the product of a real, positive projection operator times a Feynman scalar propagator. The Dirac propagator must contribute with the same sign to the imaginary part as does the scalar propagator. Our only concern—still assuming the spin-0 loop is well behaved—is the overall external sign of the amplitude.

The "negative energy" projection operator enters when we consider an antiparticle propagating forward in time as a negative energy particle propagating backward in time, as Feynman does in his prescription for the loop amplitude. We need to examine the particle propagator Λ_+, with momentum p continued to the reversed four-momentum $-p$. We find that it is *not* directly the (more properly called) antiparticle projection operator $\Lambda_-(p)$. An extra minus sign is needed in the continuation

$$\Lambda_-(p)=\Lambda_+(-p)=\frac{-p\gamma+m}{2m}\neq\sum_s u(-p;s)\bar{u}(-p;s)$$

$$\text{but } =-\sum_s u(-p;s)\bar{u}(-p;s)$$

$$\text{or } =-\sum_s v(p;s)\bar{v}(p;s). \quad (A9)$$

An extra minus sign must be inserted in the continuation

$$u(p)\bar{u}(p)\rightarrow -u(-p)\bar{u}(-p)=-v(p)\bar{v}(p). \quad (A10)$$

The Feynman Green's function becomes

$$S_F(x'-x)=-i\Theta(t'-t)\sum \psi\bar{\psi}$$

$$\rightarrow -i\Theta(t'-t)\sum_+ \psi\bar{\psi}+i\Theta(t-t')\sum_- \psi\bar{\psi}. \quad (A11)$$

Recall that the negative energy spinors have $\bar{\psi}_-\psi_-=\bar{v}v=-1$ and require a minus sign in the projection operator $\Lambda_-(p)$.

A nearly identical situation arises in the Feynman Green's function for the spin-0 Klein–Gordon equation. Here a sign change sneaks in because the norm of positive energy states is

$$\sim\int \phi_+^\dagger(i\overleftrightarrow{\partial_0})\phi_+,$$

but the negative of this expression for negative energy states. Bjorken and Drell have a full description of the two cases in their "Relativistic Quantum Mechanics," pages 95 and 188.[43]

This lack of simple continuability is also obvious from the gap between the static $+m$ and $-m$ four-component Dirac spinors. We have (in an abbreviated 2×2 notation)

$$u(m)=\begin{pmatrix}1\\0\end{pmatrix},\quad u(-m)\equiv v(m)=\begin{pmatrix}0\\1\end{pmatrix}, \quad (A12)$$

and the projection operators

$$\Lambda_+(m)=\begin{pmatrix}1&0\\0&0\end{pmatrix}=\sum_m u(m)\bar{u}(m)$$
$$=\begin{pmatrix}1\\0\end{pmatrix}(1\quad 0)\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \quad (A13)$$

but

$$\Lambda_-(m)=\begin{pmatrix}0&0\\0&1\end{pmatrix}=-\sum_m u(-m)\bar{u}(-m)$$
$$=-\sum_m v(m)\bar{v}(m)$$
$$=-\begin{pmatrix}0\\1\end{pmatrix}(0\quad 1)\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \quad (A14)$$

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