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PHOTON DISTRIBUTION IN NONLINEAR COHERENT STATES

R. López–Peña,[1][1] V. I. Man'ko,[2] G. Marmo,[1] E. C. G. Sudarshan,[3] and F. Zaccaria[1]

[1]*Dipartimento di Scienze Fisiche, Universitá "Federico II" di Napoli and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia, Napoli 80126, Italia*
[2]*P. N. Lebedev Physical Institute, Russian Academy of Sciences, Leninskii Pr. 53, Moscow 117924, Russia*
[3]*Physics Department, Center for Particle Physics, University of Texas, Austin, Texas 78712*

e-mail: manko@sci.lebedev.ru gimarmo@na.infn.it sudarshan@physics.utexas.edu zaccaria@na.infn.it

Abstract

The notion of f-oscillators generalizing q-oscillators is discussed. For the classical and quantum cases, an interpretation of the f-oscillator is provided as corresponding to a special nonlinearity of vibration for which the frequency of the oscillation depends on the energy. The f-coherent states generalizing the q-coherent states are constructed. Applied to quantum optics, the photon distribution function and photon number means and dispersions are calculated for the f-coherent states as well as the Wigner–Moyal function and Q-function. As an example, it is shown how this nonlinearity may affect the Planck's distribution formula.

1. Introduction

Recently the nonlinear coherent states were discussed for trapped ions in [1]. The nonlinear coherent states are natural generalizations of the coherent states of the linear harmonic oscillator to incorporate possible nonlinearity of the vibrations. Another physical example of the nonlinear oscillator are the light modes propagating in a Kerr medium modelled by a quartic nonlinearity which may transform a coherent state into a superposition of coherent states called a Schrödinger cat state [2, 3]. Different types of other interesting states were suggested in [4] and may be related to nonlinearities of field vibrations. Some of these states are particular cases of generalized coherent states introduced in [5, 6] where the superposition of number states with the intensity-dependent phase of the superposition coefficients was suggested. The general scheme of nonclassical states discussed in [7] contains also as partial cases the superposition of states such as the even and odd coherent states [8], which may be realized for trapped ions [9, 10].

The notion of coherent states [11–14], referred to as classical photon states, as the specific superposition of number states with the same quadrature dispersions as that of the vacuum state, permitted the use of language and intuition developed from the study of the classical mechanics of harmonic oscillators in order to treat their quantum counterpart. On the other hand, the notion of quantum q-oscillator [15, 16] was interpreted [17, 18] as a nonlinear oscillator with a very specific type of nonlinearity, in which the frequency of vibration depends on the energy of the vibrations through the hyperbolic cosine function containing a parameter of nonlinearity. The interpretation of q-oscillators becomes obvious if one uses the classical counterpart of the original quantum q-oscillators. This observation suggests that there might exist other types of nonlinearity for which the frequency of oscillation varies with the amplitude, in a manner different from the cosh dependence; we label this dependence by a function f. Such classical oscillators (and their quantum partners) were called f-oscillators [17–21]. Below we review the results of [19–21] and discuss the influence of nonlinearity on the photon distribution function and squeezing and correlation of the quadratures.

[1]On leave from Instituto de Ciencias Nucleares, UNAM, México.

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For stationary systems, one can consider such changes of the Hamiltonian, which is an integral of motion, that produce a new Hamiltonian which is some function of the initial one. Then if there are no degeneracies in the spectra for the initial Hamiltonian, the eigenstates of the new Hamiltonian coincide with two states of the old Hamiltonian. But for the new systems, the energy spectra are different. This produces time evolution of the phase factors of the eigenstates such that they vary with different velocities in complete analogy to the classical motion of the corresponding deformed classical systems, moving along their trajectories in the phase space with reparametrized velocities.

The extension of coherent-states constructions was studied in detail in [5] and called generalized coherent states in [6], which took into account, partially, such reparametrization, since the contribution of different number states in the new superposition contains the phases, depending on the nonlinearity. A recent analysis of the coherent and the generalized coherent states is given in [4].

There are other interesting types of nonclassical states like harmonious states [22] and correlated states [23]. The statistical properties of the quantum systems in these states are different from the statistical properties of the systems in the conventional coherent states. Thus, the photon statistics in coherent states [11–14] and in generalized coherent states [5, 6] is described by the Poisson distribution function, because the Poissonian photon statistics of conventional coherent states is not influenced by a phase factor of α in the series decomposition of the generalized coherent state. The nonlinearities deform the Poissonian statistics.

The aim of this paper is to study the statistics of the f-coherent states. We study photon distribution functions and such quasidistributions as the Wigner–Moyal function [24, 25] and the Husimi–Kano function [26, 27]. Also the quadrature means and dispersions are calculated. The squeezing and quadrature correlation in f-coherent states are discussed. The generalized correlated states [28] for the two-mode systems will be extended taking into account the f-nonlinearity.

2. f-Oscillators

To discuss the quantum f-oscillator, we start by recalling some notions about the harmonic-oscillator operators a and $a^\dagger$ whose algebraic structure is

$$[a, a^\dagger] = 1 .$$

Consider now a "distortion" of a of the form

$$A = a\, f(\hat{n}) = f(\hat{n}+1)\, a\,, \qquad \hat{n} = a^\dagger a\,. \tag{1}$$

Then

$$F(\hat{n}) \doteq [A, A^\dagger] = (\hat{n}+1) f(\hat{n}+1)\Big(f(\hat{n}+1)\Big)^\dagger - \hat{n} f(\hat{n})\Big(f(\hat{n})\Big)^\dagger ,$$

while the q-commutator is

$$G(\hat{n}) \doteq [A, A^\dagger]_q \doteq AA^\dagger - qA^\dagger A = (\hat{n}+1) f(\hat{n}+1)\Big(f(\hat{n}+1)\Big)^\dagger - q\hat{n} f(\hat{n})\Big(f(\hat{n})\Big)^\dagger . \tag{2}$$

Vice versa, if we are given the commutator or q-commutator, i.e., given the functions F or G, we obtain

$$f(n) = \frac{1}{\sqrt{n}} \left(\sum_{j=0}^{n-1} F(j) + \tilde{F} \right)^{1/2} , \qquad n \neq 0 \tag{3}$$

and

$$f(n)=\frac{1}{\sqrt{n}}\left(\sum_{j=0}^{n-1}q^{j}G(n-j-1)+\tilde{G}q^{n-1}\right)^{1/2}, \qquad n\neq 0, \tag{4}$$

respectively, with $\tilde{F}$ and $\tilde{G}$ arbitrary constants. In the case of q-oscillator operators, the function f depends also on a continuous parameter in order to obtain the harmonic-oscillator operators as a limiting case. Starting with the q-commutation relation [15, 16]

$$G(\hat{n})=q^{-\hat{n}}=e^{-\lambda\hat{n}}, \qquad \lambda=\ln q, \qquad \lambda\in\mathbb{R}, \tag{5}$$

one obtains

$$f(n)=\sqrt{\frac{1}{n}\frac{q^{n}-q^{-n}}{q-q^{-1}}}=\sqrt{\frac{\sinh\lambda n}{n\sinh\lambda}}, \tag{6}$$

setting $f(0)=1$. The phase operators [29] are actually deformations of the Bose operators of the kind we are studying and lead to the harmonious states [22]. In this case,

$$f(n)=\frac{1}{\sqrt{n}}, \tag{7}$$

so that

$$A\mid n\rangle=\mid n-1\rangle, \qquad A^{\dagger}\mid n\rangle=\mid n+1\rangle, \qquad n\neq 0, \qquad A\mid 0\rangle=0. \tag{8}$$

3. Nonlinear Coherent States

One can consider the eigenfunctions of A, $\mid\alpha,f\rangle$ in a Hilbert space. They satisfy the equation

$$A\mid\alpha,f\rangle=\alpha\mid\alpha,f\rangle, \qquad \alpha\in\mathbb{C}. \tag{9}$$

The decomposition of $\mid\alpha,f\rangle$ in the Fock space is

$$\mid\alpha,f\rangle=\sum_{n=0}^{\infty}c_{n}\mid n\rangle, \qquad c_{n}=c_{0}\frac{\alpha^{n}}{\sqrt{n!}\,[f(n)]!}, \tag{10}$$

in which

$$[f(n)]!=f(0)f(1)\cdots f(n), \qquad N_{f,\alpha}=c_{0}=\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\,|[f(n)]!|^{2}}\right)^{-1/2}. \tag{11}$$

In the coordinate representation, the wave function is

$$\psi_{\alpha,f}(x)=\pi^{-1/4}N_{\alpha,f}\,e^{-x^{2}/2}\sum_{n=0}^{\infty}\left(\frac{\alpha}{\sqrt{2}}\right)^{n}\frac{1}{n!\,[f(n)]!}H_{n}(x), \tag{12}$$

where H_n is the Hermite polynomial of degree n.

For the momentum representation, the formula is the same with $H_n(p)$ replacing $H_n(x)$.

For the Bargmann representation (the usual coherent states), the wave function $\langle z \mid \alpha, f\rangle$, where we use the basis $\mid z\rangle$, $(z \in \mathbb{C})$ with $a \mid z\rangle = z \mid z\rangle$, takes the form

$$\psi_{\alpha,f}(z) = N_{\alpha,f}\, e^{-|z|^2/2} \sum_{n=0}^{\infty} (z^*\alpha)^n \frac{1}{n!\,[f(n)]!}\,. \tag{13}$$

In the limit $f \to 1$, the usual wave function is recovered

$$\psi_{\alpha,1}(z) = \exp\left(-\frac{|z|^2 + |\alpha|^2}{2} + z^*\alpha\right). \tag{14}$$

In the Wigner–Moyal representation [24], the density matrix for the f-coherent state reads

$$\begin{aligned} W_f(x,p) &= 2\,N_{\alpha,f}^2\, e^{-(x^2+p^2)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!\,[f(m)]!}\,\frac{1}{[f(n)]!} \\ &\quad\times(-\alpha)^n \alpha^{*m} [\sqrt{2}(x-ip)]^{m-n} L_n^{m-n}\Big(2(x^2+p^2)\Big), \end{aligned} \tag{15}$$

where L_m^n denotes a generalized Laguerre polynomial. For the particular case of the q-oscillator.

$$\begin{aligned} W_{q,\alpha} &= 2\left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]!}\right)^{-1} e^{-(x^2+p^2)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{[m]!}\,\frac{n!}{[n]!} \\ &\quad\times(-\alpha)^n \alpha^{*m} [\sqrt{2}(x-ip)]^{m-n} L_n^{m-n}\Big(2(x^2+p^2)\Big). \end{aligned} \tag{16}$$

The Husimi–Kano [26, 27] Q-function of the f-coherent states is

$$Q_f(z,z^*) = e^{-|z|^2} N_{f,\alpha}^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(z^*\alpha)^m}{m!\,[f(m)]!}\,\frac{(z\alpha^*)^n}{n!\,[f(n)]!}\,. \tag{17}$$

For the harmonious states, the Wigner–Moyal function is

$$\begin{aligned} W_h(x,p) &= 2\,(1-|\alpha|^2)\, e^{-(x^2+p^2)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \sqrt{\frac{n!}{m!}} \\ &\quad\times(-\alpha)^n \alpha^{*m} \left[\sqrt{2}(x-ip)\right]^{m-n} L_n^{m-n}\Big(2(x^2+p^2)\Big), \end{aligned} \tag{18}$$

and the Husimi–Kano function is

$$Q_h(z,z^*) = e^{-|z|^2}\,(1-|\alpha|^2) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(z^*\alpha)^m}{\sqrt{m!}}\,\frac{(z\alpha^*)^n}{\sqrt{n!}}\,. \tag{19}$$

The Husimi and Wigner distributions can be rewritten in a form which shows the dependence on the phases only changing summation order. Defining

$$\begin{aligned} \alpha &= |\alpha|\,\exp(i\varphi_\alpha)\,, \\ z &= |z|\,\exp(i\varphi_z)\,, \\ \varphi_{ps} &= \arctan\left(\frac{p}{q}\right), \end{aligned}$$

we obtain for the Husimi distribution

$$
\begin{aligned}
Q_{f,\alpha}(z, z^*) &= |\mathcal{N}_{f,\alpha}|^2 \exp(-|z|^2) \left\{ \sum_{s=0}^{+\infty} \frac{(|\alpha|\,|z|)^{2s}}{\{s!\,[f(s)]!\}^2} \right. \\
&+ 2 \sum_{s=1}^{+\infty} (|\alpha|\,|z|)^s \sum_{t=0}^{[(s-1)/2]} \frac{1}{(s-t)!\,[f(s-t)]!} \frac{1}{s!\,[f(s)]!} \\
&\left. \times \cos\Big((s-2t)(\varphi_z - \varphi_\alpha)\Big) \right\},
\end{aligned}
$$

and for the Wigner distribution

$$
\begin{aligned}
W_{f,\alpha}(p, q) &= 2\,|\mathcal{N}_{f,\alpha}|^2 \exp\left(-(p^2+q^2)\right) \left\{ \sum_{s=0}^{+\infty} (-1)^s\, |\alpha|^{2s}\, \{s!\,[f(s)]!\}^{-2}\, L_s^0\Big(2(p^2+q^2)\Big) \right. \\
&+ 2 \sum_{s=1}^{+\infty} |\alpha|^s \sum_{t=0}^{(s-1)/2} (-1)^t \left\{2(p^2+q^2)\right\}^{(s/2)-t} \{(s-t)!\,[f(s-t)]!\,s!\,[f(s)]!\}^{-1} \\
&\left. \times \cos\Big((s-2t)(\varphi_{ps} - \varphi_\alpha)\Big)\, L_t^{s-2t}\Big(2(p^2+q^2)\Big) \right\}.
\end{aligned}
$$

The photon distribution in the f-coherent state is

$$
P_{f,\alpha}(n) = \left(\sum_{j=0}^{\infty} \frac{|\alpha|^{2j}}{j!\,([f(j)]!)^2} \right)^{-1} \frac{|\alpha|^{2n}}{n!\,([f(n)]!)^2}. \tag{20}
$$

Then for the photon distribution in the q-coherent state, we have

$$
P_{f_q,\alpha}(n) = \left(\sum_{j=0}^{\infty} \frac{|\alpha|^{2j}}{\left(\dfrac{\sinh \lambda j}{\sinh \lambda}\right)!} \right)^{-1} \frac{|\alpha|^{2n}}{\left(\dfrac{\sinh \lambda n}{\sinh \lambda}\right)!}. \tag{21}
$$

The property of this distribution is that, for large n ($n \gg \lambda^{-1}$), the probability to have n photons differs substantially from the Poissonian distribution due to the exponentially decreasing effect of the denominator. The mean photon number $\langle n \rangle_q$ is given for the q-nonlinear field by

$$
\langle n \rangle_q = \left(\sum_{j=0}^{\infty} \frac{|\alpha|^{2j}}{\left(\dfrac{\sinh \lambda j}{\sinh \lambda}\right)!} \right)^{-1} \sum_{n=0}^{\infty} \frac{n\,|\alpha|^{2n}}{\left(\dfrac{\sinh \lambda n}{\sinh \lambda}\right)!}; \tag{22}
$$

and the second moment by

$$
\langle n^2 \rangle_q = \left(\sum_{j=0}^{\infty} \frac{|\alpha|^{2j}}{\left(\dfrac{\sinh \lambda j}{\sinh \lambda}\right)!} \right)^{-1} \sum_{n=0}^{\infty} \frac{n^2|\alpha|^{2n}}{\left(\dfrac{\sinh \lambda n}{\sinh \lambda}\right)!}. \tag{23}
$$

The dispersion $\sigma_{q,n}^2 = \langle n^2 \rangle_q - \langle n \rangle_q^2$ depends, of course, on the nonlinearity parameter λ.

4. Quadrature Dispersions

Now we calculate the squeezing and correlation of the quadrature components in the f-coherent states introduced.

For quadrature dispersion $\sigma_x^2 = \langle\alpha, f \mid x^2 \mid \alpha, f\rangle - \langle\alpha, f \mid x \mid \alpha, f\rangle^2$, we have

$$\sigma_x^2 = \frac{1}{2} + \mu_x\alpha^2 + \mu_x^*\alpha^{*2} + \nu_x\alpha\alpha^*. \tag{24}$$

with

$$\mu_x = \frac{1}{2}N_{\alpha,f}^2 \left\{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)f(n+2)n!\{[f(n)]!\}^2} - N_{\alpha,f}^2\Delta\right\} \tag{25}$$

and

$$\nu_x = N_{\alpha,f}^2 \left\{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f^2(n+1)n!\{[f(n)]!\}^2} - N_{\alpha,f}^2\Delta\right\}, \tag{26}$$

where

$$\Delta = \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)n!\{[f(n)]!\}^2}\right)^2. \tag{27}$$

For the other quadrature dispersion $\sigma_p^2 = \langle\alpha, f \mid p^2 \mid \alpha, f\rangle - \langle\alpha, f \mid p \mid \alpha, f\rangle^2$, we have

$$\sigma_p^2 = \frac{1}{2} + \mu_p\alpha^2 + \mu_p^*\alpha^{*2} + \nu_p\alpha\alpha^*\,. \tag{28}$$

with

$$\mu_p = -\frac{1}{2}N_{\alpha,f}^2 \left\{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)f(n+2)n!\{[f(n)]!\}^2} - N_{\alpha,f}^2\Delta\right\} \tag{29}$$

and

$$\nu_p = N_{\alpha,f}^2 \left\{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f^2(n+1)n!\{[f(n)]!\}^2} - N_{\alpha,f}^2\Delta\right\}. \tag{30}$$

Depending on the function $f(n)$ the dispersion σ_x^2 (σ_p^2) may become less than $1/2$. This means squeezing. One can calculate the correlation of the quadrature components in f-coherent states as

$$\sigma_{xp}^2 = \frac{\alpha^2 - \alpha^{*2}}{2i}\,N_{\alpha,f}^2 \left[\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{f(n+1)f(n+2)n!\{[f(n)]!\}^2} - \Delta\,N_{\alpha,f}^2\right]. \tag{31}$$

Then the quadrature correlation coefficient is not equal to zero. Thus, the f-coherent state has the property of being a correlated state [23]. The invariant $\sigma_x\sigma_p - \sigma_{xp}^2$ is larger than $1/4$. Thus, the f-coherent states do not minimize the Schrödinger uncertainty relation [30, 31].

Let us write

$$\begin{aligned} A &= a\,f(\hat{n}) = f(\hat{n}+1)\,a\,, \\ A^\dagger &= f(\hat{n})\,a^\dagger = a^\dagger\,f(\hat{n}+1)\,. \end{aligned}$$

where

$$\hat{n} = a^{\dagger}a\,.$$

Then we have

$$A^{\dagger}A = \hat{n}\,f^2(\hat{n})\,. \tag{32}$$

Defining the function

$$G(x) = x\,f^2(x)\,, \tag{33}$$

we see that

$$\hat{n} = G^{-1}(A^{\dagger}A)\,. \tag{34}$$

where G^{-1} is the inverse function of G, i.e.,

$$G^{-1}\Big(G(x)\Big) = x\,,$$

therefore, f is assumed to make G invertible.

We can define a q-commutator through

$$\Big[A,\,A^{\dagger}\Big]_q = AA^{\dagger} - q\,A^{\dagger}A\,. \tag{35}$$

In this case, defining the function

$$F_q(x) = G(x+1) - q\,G(x)\,, \tag{36}$$

we can write the q-commutator between A and $A^{\dagger}$ as

$$\Big[A,\,A^{\dagger}\Big]_q = F_q\Big(G^{-1}\left(A^{\dagger}A\right)\Big)\,. \tag{37}$$

Due to this commutation relation, we obtain for any function H

$$H\left(A^{\dagger}A\right)\,A^{\dagger} = A^{\dagger}\,H\Big(\left\{q\mathbb{I} + F_q\circ G^{-1}\right\}A^{\dagger}A\Big)\,. \tag{38}$$

In the same way as in the preceding section, we can get

$$\begin{aligned}
\Big[A^{\dagger}A,\,A\Big]_p &= \Big(\Big\{\mathbb{I} - p\left\{q\mathbb{I} + F_q\circ G^{-1}\right\}\Big\}(A^{\dagger}A)\Big)\,A\,,\\
\Big[A^{\dagger}A,\,A^{\dagger}\Big]_p &= \Big(\Big\{\mathbb{I} - p\left\{q\mathbb{I} + F_q\circ G^{-1}\right\}^{-1}\Big\}(A^{\dagger}A)\Big)\,A^{\dagger},
\end{aligned}$$

where we used the identity

$$(1-A)\,A^{-1} = -\Big(1-(1-A)^{-1}\Big)^{-1} \tag{39}$$

to obtain the second expression.

We have the consistency relation

$$\Big[G^{-1}\circ\left\{q\mathbb{I} + F_q\circ G^{-1}\right\}^{-1}\Big](x) + 1 = G^{-1}(x)\,. \tag{40}$$

This equality holds at $f(x) = 1$. In this case,

$$\begin{aligned} G(x) &= x\,. \\ F_q(x) &= (1-q)x + 1\,. \end{aligned}$$

The consistency relation can also be written as

$$G\Big(G^{-1}(x) - 1\Big) = \Big\{q\mathbb{I} + F_q \circ G^{-1}\Big\}^{-1}(x)\,. \tag{41}$$

The expressions above are useful because they allow us to write the dispersion for the number operator in the f-coherent states in closed form. For the expected value of the number operator, we have

$$\langle \hat{n} \rangle_{\alpha f} = \frac{|\alpha|^2}{f^2\left(G^{-1}(|\alpha|^2) + 1\right)}\,. \tag{42}$$

Meanwhile, one can obtain for the photon dispersion

$$(\Delta \hat{n})^2_{\alpha f} = \langle \hat{n}^2 \rangle_{\alpha f} - \langle \hat{n} \rangle^2_{\alpha f} \tag{43}$$

a complicated expression in terms of the functions G and F_q.

5. Two-Mode f-Coherent States

For the usual multimode harmonic oscillator, there exist generalized correlated states [28] in which the mode quadratures are statistically dependent. The quasidistributions for these states have Gaussian form and the photon distribution function is expressed in terms of multivariable Hermite polynomials [32]. It is possible to extend nontrivially the construction of one-mode f-oscillator to many modes. In particular, for two-mode states we consider the two operators

$$A_i = f(\hat{n})a_i\,, \qquad i = 1, 2\,, \tag{44}$$

where

$$\hat{n} = \hat{n}_1 + \hat{n}_2\,, \qquad \hat{n}_i = a_i^\dagger a_i \tag{45}$$

(the operators a_i satisfy boson commutation relations). These operators commute and for this reason we can construct algebraically the two-mode f-coherent state $\mid \alpha_1, \alpha_2, f\rangle$ defined by the following equations:

$$A_i \mid \alpha_1, \alpha_2, f\rangle = \alpha_i \mid \alpha_1, \alpha_2, f\rangle\,, \qquad i = 1, 2\,. \tag{46}$$

Consider the series expansion

$$\mid \alpha_1, \alpha_2, f\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} c_{n_1 n_2} \mid n_1 n_2\rangle\,, \tag{47}$$

where the Fock states $\mid n_1, n_2\rangle$ satisfy

$$a_1^\dagger a_1 \mid n_1 n_2\rangle = n_1 \mid n_1, n_2\rangle\,, \qquad n_1 \in \mathbb{Z}^\dagger \tag{48}$$

and

$$a_2^\dagger a_2 \mid n_1, n_2\rangle = n_2 \mid n_1, n_2\rangle\,, \qquad n_2 \in \mathbb{Z}^\dagger\,. \tag{49}$$

The solution of the recurrence relation obtained for the $c_{n_1 n_2}$'s is

$$c_{n_1 n_2} = c_{00}\,\frac{\alpha_1^{n_1}\alpha_2^{n_2}}{\sqrt{n_1! n_2!}[f(n)]!}\,. \tag{50}$$

with c_{00}, fixed as before by normalization, being

$$c_{00} = \left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{|\alpha_1|^{2n_1}|\alpha_2|^{2n_2}}{([f(n)]!)^2 n_1! n_2!}\right)^{-1/2}\,. \tag{51}$$

The two-mode f-coherent state can be now defined as

$$\mid \alpha_1, \alpha_2, f\rangle = c_{00}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{\alpha_1^{n_1}\alpha_2^{n_2}}{\sqrt{n_1! n_2!}\,[f(n)]!}\mid n_1, n_2\rangle\,, \qquad \alpha_i \in \mathbb{C}\,, \quad i = 1, 2\,. \tag{52}$$

It should be noted that in this form there is a coupling between the two modes, as there is a dependence of each of them on the total energy, and this interaction between the two modes in general is nonlinear. However, any homogeneous linear canonical transformation in a_i induces the same between A_i.

Another generalization for two-mode coherent states is of course obtained by means of the product of two one-mode f-coherent states of the previous section, so that there is no interaction between the two modes. After defining

$$A_i' = a_i f_i(n_i)\,, \qquad i = 1, 2\,, \tag{53}$$

and finding their eigenstates as in the previous section, we can make the tensor product obtaining

$$\begin{aligned}\mid \alpha_1, \alpha_2, f_1, f_2\rangle &= \left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{|\alpha_1|^{2n_1}|\alpha_2|^{2n_2}}{n_1! n_2!\,([f_1(n_1)]!\,[f_2(n_2)]!)^2}\right)^{-1/2}\\ &\quad\times\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{\alpha_1^{n_1}\alpha_2^{n_2}}{\sqrt{n_1! n_2!}\,[f_1(n_1)]!\,[f_2(n_2)]!}\mid n_1, n_2\rangle\,.\end{aligned} \tag{54}$$

In the case

$$f_1 = f_2 = 1\,, \tag{55}$$

we have the usual two-mode coherent states, namely,

$$\mid \alpha_1, \alpha_2, 1, 1\rangle = e^{-(|\alpha_1|^2+|\alpha_2|^2)/2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{\alpha_1^{n_1}\alpha_2^{n_2}}{\sqrt{n_1! n_2!}}\mid n_1, n_2\rangle\,. \tag{56}$$

6. Physical Application

We will discuss what physical consequences may be found if the considered f-nonlinearity influences the vibrations of the real field-mode oscillators such as, for example, electromagnetic-field oscillators or the oscillations of the nuclei in polyatomic molecules. Calculating the partition function for small q-nonlinearity parameter we have also the following q-deformed Planck's distribution formula:

$$\langle n\rangle = \frac{1}{e^{\hbar\omega/kT}-1} - \lambda^2\,\frac{\hbar\omega}{kT}\,\frac{e^{3\hbar\omega/kT}+4\,e^{2\hbar\omega/kT}+e^{\hbar\omega/kT}}{(e^{\hbar\omega/kT}-1)^4}. \tag{57}$$

This means that q-nonlinearity deforms the mean photon numbers in blackbody radiation [17]. One can write down the low- and high-temperature approximations for the deformed Planck's distribution formula [33]. For low temperatures, the behavior of the deformed Planck's distribution differs from the usual one

$$\langle n\rangle - \bar n_0 = -\lambda^2\,\frac{\hbar\omega}{kT}\,e^{-\hbar\omega/kT}. \tag{58}$$

For high temperatures, the nonlinear correction to the usual mean photon number also depends on temperature

$$\langle n\rangle - \bar n_0 = -\frac{\lambda^2}{(\hbar\omega/kT)^3}. \tag{59}$$

In both limits, we find that the mean photon number is reduced and the spectral distribution is distorted. At low temperatures, the distortion is maximum at $\hbar\omega = kT$.

As is seen, the discussed q-nonlinearity produces a correction to the Planck's mean-oscillator-energy-distribution formula, and this may also be subjected to an experimental test. As was suggested in [18] the q-nonlinearity of the field vibrations produces a blue shift effect, which is the effect of frequency increase with field intensity. For small nonlinearity parameter λ and large number of photons n in a given mode, the relative shift of the light frequency is

$$\frac{\delta\omega}{\omega} = \frac{\lambda^2}{2}\,n^2.$$

This consequence of the possible existence of a q-nonlinearity may be relevant for models of the early stage of the Universe. Another possible phenomenon related to the q-nonlinearity was considered in [34] where it was shown that, if one deforms the electrostatics equation using the method of deformed creation and annihilation operators, a point charge acquires a form factor due to q-nonlinearity.

7. Conclusion

Starting with the example of harmonic oscillator we have exhibited a family of associated nonlinear systems which are completely integrable, both in classical and quantum physics. We have shown that q-nonlinearity, associated with quantum groups, is a subcase of a more general class of possible nonlinearities. The studied nonlinearities, if they exist [35], for the electromagnetic field or for the gluons, may influence the particle decays, correlations in particle multiplicities, and a change in the Hanbury Brown–Twiss experiment results.

In the one-mode case,

$$A_f = af(a), \qquad A_f^\dagger = f(n)a^\dagger, \qquad \text{and} \qquad \left(A_f^\dagger A_f\right)^2 = n^2 f^4(n).$$

which gives

$$\langle\left(A_f^\dagger A_f\right)^2\rangle - \langle A_f^\dagger A_f\rangle^2 = \langle n^2 f^4(n)\rangle - \langle n f^2(n)\rangle^2.$$

One can see that the nonlinearity function $f(n)$ contributes to the value of the intensity dispersion.

For the two-mode case (the Hanbury Brown–Twiss effect), one has for the correlation of the intensities

$$\langle A_{1f}^{\dagger}A_{1f}A_{2f}^{\dagger}A_{2f}\rangle - \langle A_{1f}^{\dagger}A_{1f}\rangle\langle A_{2f}^{\dagger}A_{2f}\rangle = \langle n_1^2 n_2^2 f^4(n_1+n_2)\rangle - \langle n_1^2 f(n_1+n_2)\rangle\langle n_2^2 f(n_1+n_2)\rangle = \Delta\,(I_1 I_2)\,.$$

Also for the amplitude correlation, one has

$$\langle A_{1f}^{\dagger}A_{2f}\rangle = \langle f(n_1+n_2)a_1^{\dagger}a_2 f(n_1+n_2)\rangle$$

and

$$\langle f(n_1+n_2)a_1^{\dagger}a_2 f(n_1+n_2)\rangle\langle f(n_1+n_2)a_2^{\dagger}a_1 f(n_1+n_2)\rangle = \Delta_{am}\Delta_{am}^{*}\,.$$

The above formulas demonstrate the influence of f-nonlinearity on the two-mode correlations. It would naturally affect the stimulated emission rates and hence the radiative equilibrium in the presence of matter.

Some recent discussions of f-oscillators and deformed commutation relations [36, 37] were done in connection with constructing nonlinear even and odd Schrödinger cat states [38]. It has also been shown that f-coherent states can describe some other known nonclassical states [39–42]. The f-coherent states are useful in the treatment of nonlinear coherent states for a trapped ion (see [1] and [43–46]). To conclude, the particle-number stastistics of nonlinear f-oscillator states can be detected in different types of experiments both in quantum optics and atom optics.

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